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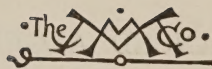


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ADVANCED ALGEBRA

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BY

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PREFACE

Most teachers of mathematics agree that a number of the topics ordinarily taught to classes in advanced algebra may be omitted without injury to the course. Some of these topics, such as multiple roots, Sturm's theorem, etc., can be more satisfactorily taken up after the student is familiar with calculus, while others, such as recurring series, continued fractions, etc., are so seldom applied in higher mathematics that they may be entirely omitted.

In accordance with this view, the College Entrance Examination Board has considerably reduced the number of topics required in advanced algebra. All subjects no longer required for the examinations of this Board are omitted from the regular course of this book, with the exception of inequalities, which is retained, since familiarity with the symbols of inequality seems to be necessary for future work. If, however, a subject appears too important for entire omission, it is placed in the Appendix. This is done in the case of indeterminate equations, logarithms, summation of series, and some other subjects.

On the other hand, graphical methods are emphasized more than is usual in text-books of this grade. The graphical method for solving cubics given in Section 578 is not met with in any other text-book, and the method for representing a cubic function by means of one standard curve (Section 583) is entirely new. Summation of series is also treated in a novel manner (Appendix IX). While the method given is almost identical with that used in many text-books, it is here presented in a more practical form, which makes it applicable to all cases.

The first twenty-two chapters are identical with the author's "Elementary Algebra," whose general plan and scope are stated in its preface as follows:

"The author has aimed to make this treatment of elementary algebra simple and practical, without, however, sacrificing scientific accuracy and thoroughness.

"Particular care has been bestowed upon those chapters which in the customary courses offer the greatest difficulties to the beginner, especially problems and factoring. The presentation of problems as given in Chapter V will be found to be quite a departure from the customary way of treating the subject, and it is hoped that this treatment will materially diminish the difficulty of this topic for young students.

"In factoring, instead of the usual multiplicity of cases, comparatively few methods are given, but these few are treated thoroughly. The cross-product method for factoring quadratic trinomials has been simplified by considering the common monomial factors (§ 116, 4); and in this form the method seemed to be preferable to the other prevailing methods. The criticism that the cross-product method is based upon guessing has no value, since all other devices are equally based upon guessing; in fact, these methods have to be empirical until quadratic equations furnish a scientific means of factoring.

"Applications taken from geometry, physics, and commercial life are numerous, but care has been taken not to introduce illustrations so complex as to require the expenditure of time for the teaching of physics or geometry. In cases, however, in which a physical or geometric formula produced an example equally good as the putting together of symbols at random, the formula has been used, as in numerical substitution, proportion, literal equations, etc.

"The book is designed to meet the requirements for admission to our best universities and colleges, in particular the requirements of the College Entrance Examination Board. This made it necessary to introduce the theory of proportions and

graphical methods into the first year's work, an innovation which seems to mark a distinct gain from the pedagogical point of view.

"By studying proportions during the first year's work, the student will be able to utilize this knowledge where it is most needed, viz. in geometry ; while in the usual course proportions are studied a long time after their principal application.

"Graphical methods have not only a great practical value, but they unquestionably furnish a very good antidote against 'the tendency of school algebra to degenerate into a mechanical application of memorized rules.' This topic has been represented in a simple, elementary way, and it is hoped that some of the modes of representation given will be considered improvements upon the prevailing methods, *e.g.* the finding of roots to several decimal places (§ 305) and the solution of quadratic equations (§ 330). The entire work in graphical methods has been so arranged that teachers who wish a shorter course may omit these chapters."

The author desires to acknowledge his indebtedness to Messrs. William P. Manguse and B. A. Heydrick for the careful reading of the proofs and for many valuable suggestions.

ARTHUR SCHULTZE.

NEW YORK,
October, 1905.

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ELEMENTARY ALGEBRA

CHAPTER I

INTRODUCTION

1. **Algebra** treats of numbers as does arithmetic, but for reasons which will appear later, numbers are frequently denoted by letters; as, a , m , x .

2. **Known numbers** are usually represented by the first letters of the alphabet; as, a , b , c .

Unknown numbers are usually represented by the last letters of the alphabet; as, x , y , z .

3. The **signs of addition, subtraction, multiplication, division, and equality** have the same meaning in algebra as they have in arithmetic.

4. A **problem** is a question proposed for solution.

5. An **equation** is a statement expressing the equality of two quantities; as, $7x = 56$.

ALGEBRAIC SOLUTION OF PROBLEMS

6. *Problem.* The sum of two numbers is 56, and the greater is six times the smaller. Find the numbers.

Let x = the smaller number.

Then $6x$ = the greater number,

and $7x$ = the sum of the two numbers.

Therefore, $7x = 56$,

$x = 8$, the smaller number.

and $6x = 48$, the greater number.

20. Two men start at the same time to travel, one from A to B , the other from B to A , two stations 66 miles apart. If the first man travels twice as fast as the second, how many miles will he travel before he meets the other?

NEGATIVE NUMBERS

EXERCISE 2

1. Subtract 10 from 17.
 2. Can 10 be subtracted from 7?
 3. In arithmetic why cannot 10 be subtracted from 7?
 4. The temperature at noon is 17° and at 4 P.M. it is 10° less. What is the temperature at 4 P.M.? State this as an example of subtraction.
 5. The temperature at 4 P.M. is 7° , and at 10 P.M. it is 10° less. What is the temperature at 10 P.M.?
 6. Do you know of any other way of expressing the last answer (3° below zero)?
 7. What then is $7 - 10$?
 8. Can you think of any other practical examples which require the subtraction of a greater number from a smaller one?
-

8. Many practical examples require the subtraction of a greater number from a smaller one, and in order to express in a convenient form the results of these, and similar examples, it becomes necessary to enlarge our concept of number, so as to include numbers less than zero.

9. **Negative numbers** are numbers smaller than zero; they are denoted by a prefixed minus sign; as -5 (read "minus 5"). Numbers greater than zero, for the sake of distinction, are frequently called positive numbers, and are written either with a prefixed plus sign, or without any prefixed sign; as $+5$ or 5 .

The fact that a thermometer falling 10° from 7° indicates 3° below zero may now be expressed

$$7^{\circ} - 10^{\circ} = -3^{\circ}.$$

Instead of saying a gain of \$30, and a loss of \$90 is equal to a loss of \$60, we may write

$$\$30 - \$90 = -\$60.$$

10. The **absolute value** of a number is the number taken without regard to its sign.

The absolute value of -5 is 5, of $+3$ is 3.

11. It is convenient for many discussions, to represent the positive numbers by a succession of equal distances laid off on a line from a point 0, and the negative numbers by a similar series in the opposite direction.

In the annexed diagram, *e.g.* an addition of 3 is equivalent to an upward motion of three spaces; hence 3 added to -4 equals -1 , 3 added to -1 equals $+2$, etc.

Similarly a subtraction of 2 is equal to a downward motion of two spaces. Hence 2 subtracted from 0 equals -2 , 2 subtracted from -2 equals -4 , etc.

EXERCISE 3

1. If in financial transactions we indicate a man's income by a positive sign, what does a negative sign indicate?
2. State in what manner the positive and negative signs may be used to indicate north and south latitude, east and west longitude, motion upstream and downstream.
3. If south latitude is indicated by a positive sign, by what is north latitude represented?
4. What is the meaning of -5° north latitude? of the year -50 A.D.? of an easterly motion of -6 yards per second?
5. If the temperature at 4 A.M. is -7° and at 9 A.M. it is 6° higher, what is the temperature at 9 A.M.? What, therefore, is $-7 + 6$?

6. A vessel starts from a point in 25° north latitude, and sails 38° due south. (a) Find the latitude at the end of the journey. (b) Find $25 - 38$.

7. A vessel starts from a point in 25° south latitude, and sails 12° due south. (a) Find the latitude at the end of the journey. (b) Subtract 12 from -25 .

8. From 20 subtract 30.

18. To -1 add 2.

9. From 4 subtract 6.

19. From 1 subtract 2.

10. From 7 subtract 8.

20. To -7 add 8.

11. From 19 subtract 24.

21. To -7 add 2.

12. From 0 subtract 12.

22. From -1 subtract 2.

13. From -12 subtract 10.

23. Add -1 and 2.

14. From -2 subtract 4.

Which is the greater number:

15. From -1 subtract 1.

24. 1 or -1 ?

16. To -6 add 12.

25. -1 or -3 ?

17. To -2 add 1.

26. -3 or -4 ?

27. By how much is -9 greater than -11 ?

28. What is the difference in temperature between 20° above zero and 20° below zero?

29. What is the difference between $+20$ and -20 ?

NUMBERS REPRESENTED BY LETTERS

12. For many purposes of arithmetic it is advantageous to express numbers by letters.

I. By denoting an unknown number by x , it becomes possible to perform arithmetical operations with unknown quantities, and to solve problems thereby (§ 6).

II. By such a use of letters, arithmetical rules and principles may be expressed very concisely, *e.g.* if we wish to express

briefly the principle: The square root of a fraction is equal to the square root of its numerator divided by the square root of its denominator, we may write

$$\sqrt{\frac{\text{numerator}}{\text{denominator}}} = \frac{\sqrt{\text{numerator}}}{\sqrt{\text{denominator}}}.$$

By representing the numerator by a letter, *e.g.* a , and the denominator by b , we may write more briefly

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

In this equation a and b denote any numbers whatsoever.

III. The discovery and demonstration of general arithmetical laws constitute the third advantage derived from the employment of letters.

When each of two numbers is multiplied by itself, and the smaller product is subtracted from the greater, the answer is the same as if we multiply the sum of the two numbers by their difference.

$$E.g. \quad (14 \times 14) - (13 \times 13) = 196 - 169 = 27.$$

$$(14 + 13) \times (14 - 13) = 27 \times 1 = 27.$$

If we try a great many arithmetical numbers, we find the principle always correct, but this does not prove that it is correct for all numbers. In algebra, however, we can easily show that $(a \times a) - (b \times b) = (a + b) \times (a - b)$ (§ 65), and since a and b denote any number whatsoever, the law is correct for any two numbers.

IV. For a fourth advantage, see § 30.

EXERCISE 4

1. If the letter a means 1000, what is the value of $6a$?
2. What is the value of $6a$ if $a = 7$? if $a = \frac{1}{2}$?
3. If a boy has $6d$ marbles and wins $4d$ marbles, how many marbles has he?

4. Is the last answer correct for any value of d ?
5. What is the sum of $9b$ and $6b$?
6. Find the numerical value of the last answer if $b = 17$.
7. If c represents a certain number, what represents 10 times that number?
8. From $24m$ subtract $15m$.
9. What is the numerical value of the last answer if $m = 7$?
10. From $12m$ subtract $15m$, and find the numerical value of the answer if $m = 12$.
11. Add $12p$, $2p$, $7p$, and subtract $24p$ from the sum.
12. From $-12q$ subtract $20q$.
13. Add $-12q$ and $+12q$.
- ✓ 14. From 0 subtract $12x$.
15. Add $-3x$ and $6x$.
16. From $-12x$ subtract 0.
17. From $-12p$ subtract $12p$.
18. If $a = 100$, then $7a = 700$. What sign, therefore, is understood between 7 and a in the expression $7a$?

FACTORS, POWERS, AND ROOTS

13. If there is no sign between two letters, or a letter and a number, a sign of multiplication is understood.

$5 \times a$ is generally written $5a$, $m \times n$ is written mn .

Between two figures, however, a sign of multiplication (either \times or \cdot) has to be employed; as, 4×7 , or $4 \cdot 7$.

4×7 cannot be written 47, for 47 means $40 + 7$.

14. A **product** is the result obtained by multiplying together two or more quantities, each of which is a factor of the product.

Since $24 = 3 \times 8$, or 12×2 , each of these numbers is a factor of 24.

Similarly, 7, a , b , and c are factors of $7abc$.

15. A **power** is the product of two or more equal factors; thus, $aaaaa$ is called the "fifth power of a ," and written a^5 ; $aaaaaa$, or a^6 , is "the 6th power of a ," or a 6th.

The *second power* is also called the *square*, and the *third power* the *cube*; thus, 12^2 (read "12 square") equals 144.

16. The **base** of a power is the number which is repeated as a factor.

The base of a^3 is a .

17. An **exponent** is the number which indicates how many times a base is to be used as a factor. It is placed a little above and to the right of the base.

The exponent of m^6 is 6; n is the exponent of a^n .

EXERCISE 5

1. Write and find the numerical value of the square of 6, the cube of 7, the fourth power of 2.

Find the numerical values of the following powers:

$$2. 6^2. \quad 5. 1^{20}. \quad 8. 0^5. \quad 11. 20^1. \quad 14. 1.2^2.$$

$$3. 2^6. \quad 6. 2^3. \quad 9. (2\frac{1}{2})^2. \quad 12. (\frac{1}{2})^6. \quad 15. (\frac{1}{3})^3.$$

$$4. 10^5. \quad 7. 3^2. \quad 10. 3.5^2. \quad 13. .001^3.$$

If $a = 2$, $b = 3$, $c = \frac{1}{4}$, and $d = 1$, find the numerical values of:

$$16. a^3. \quad 18. c^2. \quad 20. d^{12}. \quad 22. 5^b. \quad 24. a^b.$$

$$17. b^2. \quad 19. d^6. \quad 21. 5^a. \quad 23. 5^d. \quad 25. b^a.$$

26. The distance of the north pole from the equator is 10^7 meters. Find the distance in meters.

27. A kilometer equals 10 hectometers, 1 hectometer = 100 meters, 1 meter = 100 centimeters, and 1 centimeter = 10 millimeters.

Express as a power of 10 the number of

- (a) Millimeters contained in a meter.
- (b) Millimeters contained in a kilometer.
- (c) Square millimeters contained in a square kilometer.

28. Express as a power of 2 the number of parents, grandparents, great-grandparents, etc., a person may have.

29. What exponent is understood when none is written, as in x or y ?

18. In a product any factor is called the **coefficient** of the product of the other factors.

In $12mn^3p$, 12 is the coefficient of mn^3p , 12 m is the coefficient of n^3p .

19. A **numerical coefficient** is a coefficient expressed entirely in figures.

In $-17xyz$, -17 is the numerical coefficient.

When a product contains no numerical coefficient, 1 is understood; thus $a = 1a$, $a^3b = 1a^3b$.

20. When several powers are multiplied, the beginner should remember that every exponent refers only to the number near which it is placed.

$3a^2$ means $3aa$, while $(3a)^2 = 3a \times 3a$.

$9aby^3 = 9abyyy$.

$16x^2y^3z = 16xyyyyz$.

EXERCISE 6

If $a = 2$, $b = 1$, $c = 3$, and $x = 4$, find the numerical value of:

- | | | | |
|------------|---------------|-------------|-----------------|
| 1. $3a$. | 4. a^3 . | 7. $3a^3$. | 10. $7ab^2$. |
| 2. abc . | 5. b^{20} . | 8. $5b^7$. | 11. $3abc^2$. |
| 3. $6bx$. | 6. $4abcx$. | 9. $9c^3$. | 12. $3a^2b^3$. |

If $a = 3$, $b = 2$, $c = 1$, $d = 4$, find the numerical value of:

- | | | | |
|--------------------|-----------------|------------------------|-------------------------|
| 13. $14ab^2c^3$. | 16. $10ab^2d$. | 19. 2^b . | 22. $\frac{1}{6}ab^2$. |
| 14. $7a^2b^2c^5$. | 17. a^4 . | 20. 7^c . | 23. a^b . |
| 15. $8abc^9$. | 18. 4^a . | 21. $\frac{2}{3}a^3$. | 24. b^a . |

If $a = 2$, $p = 3$, $q = 4$, $x = \frac{1}{2}$, $y = 0$, find the numerical value of:

- | | | |
|--------------------------|------------------|------------------------|
| 25. $3qp^2y$. | 29. x^2 . | ✓ 33. 2^ax^2 . |
| 26. a^qp^2y . | 30. x^5 . | 34. x^aq^a . |
| 27. $\frac{4ap^2}{9q}$. | 31. $4x^3$. | 35. $\frac{32}{a^q}$. |
| 28. a^3p^2 . | 32. $4a^2px^2$. | |

36. What is the numerical coefficient in each of the expressions in Exs. 1-6?

37. What are the coefficients of x in Exs. 3 and 6?

21. A **root** is one of the equal factors of a power. According to the number of equal factors, it is called a square root, a cube root, a fourth root, etc.

3 is the square root of 9, for $3^2 = 9$.

5 is the cube root of 125, for $5^3 = 125$.

a is the fifth root of a^5 , the n th root of a^n .

The n th root is indicated by the symbol $\sqrt[n]{}$; thus $\sqrt[5]{a}$ is the fifth root of a , $\sqrt[3]{27}$ is the cube root of 27, $\sqrt[2]{a}$, or more simply \sqrt{a} is the square root of a .

Using this symbol we may express the definition of root by $(\sqrt[n]{a})^n = a$.

22. The **index** of a root is the number which indicates what root is to be taken. It is written in the opening of the radical sign.

In $\sqrt[7]{a}$, 7 is the index of the root.

23. The signs of aggregation are: the *parenthesis*, (); the *bracket*, []; the *brace*, { }; and the *vinculum*, —.

They are used, as in arithmetic, to indicate that the expressions included are to be treated as a whole.

Each of the forms $10 \times (4 + 1)$, $10 \times [4 + 1]$, $10 \times \overline{4 + 1}$ indicates that 10 is to be multiplied by $4 + 1$ or by 5.

$(a - b)$ is sometimes read "quantity $a - b$."

EXERCISE 7

If $a = 2$, $b = 4$, $c = 1$, $d = 0$, $x = 9$, find the numerical value of:

- | | | |
|---------------------|--------------------------|------------------------------|
| 1. \sqrt{b} . | 7. $\sqrt{b^2}$. | 13. $3(a + b)$. |
| 2. $\sqrt[3]{4a}$. | 8. $\sqrt[3]{a^3}$. | 14. $(a + b)\sqrt{b}$. |
| 3. $\sqrt{4b}$. | 9. $a\sqrt{b}$. | 15. $[a + b^2]\sqrt[3]{c}$. |
| 4. $\sqrt[3]{3x}$. | 10. $2a\sqrt{9b}$. | 16. $(a + b)^2$. |
| 5. $\sqrt{4x}$. | 11. $4ad\sqrt[3]{54b}$. | 17. $(b + c)^3$. |
| 6. $\sqrt{9c}$. | 12. $a + b$. | |

18. In Ex. 14 what is the coefficient of the \sqrt{b} ? of $(a + b)\sqrt{b}$?

ALGEBRAIC EXPRESSIONS AND NUMERICAL SUBSTITUTIONS

24. An **algebraic expression** is a collection of algebraic symbols representing some number; e.g. $6a^2b - 7\sqrt{ac^2} + 9$.

25. A **monomial** or **term** is an expression whose parts are not separated by a sign $+$ or $-$; as $3ax^2$, $-9\sqrt{x}$, $\frac{3ab}{5c^2}$.

$a(b + c + d)$ is a monomial, since the parts are a and $(b + c + d)$.

26. A **polynomial** is an expression containing more than one term.

$4x + y$, $\frac{7x}{y} + \sqrt{z} - 3a^3b$; and $a^4 + b^4 + c^4 + d^4$ are polynomials.

27. A **binomial** is a polynomial of two terms.

$a^2 + b^2$, and $\frac{3}{4} - \sqrt[3]{a}$ are binomials.

28. A **trinomial** is a polynomial of three terms.

$a + b + c$, $a + 9b + \sqrt{3}$ are trinomials.

29. In a polynomial each term is treated as if it were contained in a parenthesis, *i.e.* each term has to be computed before the different terms are added and subtracted. Otherwise all operations of addition, subtraction, multiplication, and division are to be performed in the order in which they are written from left to right.

E.g. $3 + 4 \cdot 5$ means $3 + 20$ or 23.

Ex. 1. Find the value of $4 \cdot 2^3 + 5 \cdot 3^2 - \frac{9\sqrt{36}}{2}$.

$$\begin{aligned} & 4 \cdot 2^3 + 5 \cdot 3^2 - \frac{9\sqrt{36}}{2} \\ &= 4 \cdot 8 + 5 \cdot 9 - \frac{9 \cdot 6}{2} \\ &= 32 + 45 - 27 \\ &= 50. \end{aligned}$$

Ex. 2. If $a=5$, $b=3$, $c=2$, $d=0$, find the numerical value of $6ab^2 - 9ab^2c + \frac{2}{5}a^3b - 19a^2bcd$.

$$\begin{aligned} & 6ab^2 - 9ab^2c + \frac{2}{5}a^3b - 19a^2bcd \\ &= 6 \cdot 5 \cdot 3^2 - 9 \cdot 5 \cdot 3^2 \cdot 2 + \frac{2}{5} \cdot 5^3 \cdot 3 - 19 \cdot 5^2 \cdot 3 \cdot 2 \cdot 0 \\ &= 6 \cdot 5 \cdot 9 - 9 \cdot 5 \cdot 9 \cdot 2 + \frac{2}{5} \cdot 125 \cdot 3 - 0 \\ &= 270 - 810 + 150 \\ &= -390. \end{aligned}$$

EXERCISE 8

1. State what kind of expressions are Exs. 18-27 of this exercise.

If $a=5$, $b=2$, $c=1$, $d=0$, $x=\frac{1}{2}$, and $y=\frac{1}{3}$, find the numerical value of:

2. $a + b + 3c$.

4. $a - b$.

3. $a + 5bc$.

5. $a - 4b$.

6. $5a^2 + 7ab$.
 7. $ab - 2b^3c - 4cd$.
 8. $8abc - 20bc - 40abd$.
 9. $7a + 12d - 19c - 20b$.
 10. $5ab + 6c^2 - 4bd + 7a^2$.
 11. $4c + 6a - 8b^3 + 6x$.
 12. $4cx + 7b^2 - 2a^2 - 4x$.
 13. $a^3 + b^3 + c^3 + d^3$.
 14. $3ad - 6b^2 + 2dx + 4b^2$.
 15. $2a^2b - 6ab^2 + 7ac^2 - 9a^3d$.
 16. $x^2 + y^2$.
 17. $2x^2 + 3y^2$.
18. $(a+b)c$.
 19. $a^2 + x(b+2c)$.
 20. $a + \sqrt{a^2 + 11c}$.
 21. $2a^3 - (a+b)(c+d)$.
 22. $30xy(a+b)\sqrt{a+b+c+2x}$.
 23. $\frac{a}{5} + \frac{b}{2} + \frac{c}{3}$.
 24. $9abc + \sqrt{c} + 4d$.
 25. $\{(a+b) + c\}d - 6xy$.
 26. $\frac{ab}{2c} + \frac{4bc}{4} + \frac{3}{y}$.
 27. $\frac{x}{a} + \frac{y}{b} + \frac{d}{c}$.

28. $7c^2d^2 - x(3abx + c^2) - cx(d+b) - 12c^2x$.
 29. $\frac{a+b}{a-b}$.
 30. $\frac{1}{a} - \frac{1}{b} + \frac{1}{c}$.
 31. $\frac{a^2+7}{a^2-9}$.
 32. $\frac{6x+6y}{a^2+b^2}$.
 33. $\frac{10c+11d}{3b^2+5c^2}$.

$$34. (a+b)(c+d) - (a+c)(b+d) + (a+d)(b-c).$$

Express in algebraic symbols:

35. Six times a plus 3 times b .
 36. Six times the square of a minus five times the cube of b .
 37. Eight x cube minus six x cube plus y square.
 38. Six m cube plus four times the quantity a minus b .
 39. The quantity a plus b multiplied by the quantity a^2 minus b^2 .
 40. Twice a^3 diminished by 5 times the square root of the quantity a minus b square.
 41. Read the expressions of Exs. 2-6 of the exercise.

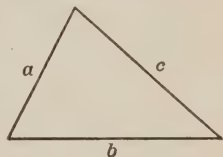
30. The representation of numbers by letters makes it possible to state very briefly and accurately some of the principles of arithmetic, geometry, physics, and other sciences.

Ex. If the three sides of a triangle contain respectively a , b , and c feet (or other units of length), and the area of the triangle is S square feet (or squares of other units selected), then

$$S = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(b-a+c)}.$$

E.g. the three sides of a triangle are respectively 13, 14, and 15 feet, then $a = 13$, $b = 14$, and $c = 15$; therefore

$$\begin{aligned} S &= \frac{1}{4}\sqrt{(13+14+15)(13+14-15)(13-14+15)(14-13+15)} \\ &= \frac{1}{4}\sqrt{42 \cdot 12 \cdot 14 \cdot 16} \\ &= \frac{1}{4} \times 336 \\ &= 84, \text{ i.e. the area of the triangle equals} \\ &\quad 84 \text{ square feet.} \end{aligned}$$



EXERCISE 9

1. By using the formula

$$S = \frac{1}{4}\sqrt{(a+b+c)(a+b-c)(a-b+c)(b-a+c)},$$

find the area of a triangle whose sides are respectively

(a) 5, 12, and 13 feet.

(c) 4, 13, and 15 meters.

(b) 3, 4, and 5 inches.

(d) 9, 10, and 17 yards.

2. If the radius of a circle is R units of length (inches, meters, etc.), the area $S = 3.1416 \cdot R^2$ square units (square inches, square meters, etc.). Find the area of a circle whose radius is

(a) 1000 meters.

(b) 3 inches.

(c) 240,000 miles.

3. If i represents the simple interest of p dollars at $r\%$ in n years, then $i = p \cdot n \cdot r\%$, or $\frac{pnr}{100}$.

Find by means of this formula :

(a) The interest on \$730 for 4 years at $2\frac{1}{2}\%$.

(b) The interest on \$380 for 2 years at 4% .

(c) The interest on \$246 for 4 months at 7% .

4. If I represents the compound interest of p dollars at $r\%$ for n years (compounded yearly), then $I = p\left(1 + \frac{r}{100}\right)^n - p$.

Find the compound interest of :

(a) \$400 for 3 years at 10% .

(b) \$1200 for 4 years at 20% .

(c) \$1 for 2 years at 5% .

5. If the diameter of a sphere equals d units of length, the surface $S = 3.1416 d^2$ (square units). (The number 3.1416 is frequently denoted by the Greek letter π . This number cannot be expressed exactly, and the value given above is only an approximation.)

Find the surface of a sphere whose diameter equals :

(a) 8000 miles.

(b) 2 inches.

(c) 12 feet.

6. If the diameter of a sphere equals d feet, then the volume

$$V = \frac{\pi D^3}{6} \text{ cubic feet.}$$

Find the volume of a sphere whose diameter equals :

(a) 10 feet.

(b) 6 feet.

(c) 11 feet.

7. A body falling from a state of rest, passes in t seconds over a space $S = \frac{1}{2} gt^2$. The value of g for New York is 32.16 feet, or 980 cm. (This formula does not take into account the resistance of the atmosphere.)

(a) How far does a body fall from a state of rest in 3 seconds?

(b) A stone dropped from the top of a tree reached the ground in $2\frac{1}{2}$ seconds. Find the height of the tree.

(c) How far does a body fall from a state of rest in $\frac{1}{10}$ of a second?

(d) On the surface of the moon, how far does a body fall from a state of rest in 3 seconds, if $g = 5.4$ ft.?

8. If F denotes the number of degrees of temperature indicated on the Fahrenheit scale, the equivalent reading C on the Centigrade scale may be found by the formula

$$C = \frac{5}{9}(F - 32).$$

Change the following readings to Centigrade readings:

(a) 100° F.

(b) 32° F.

(c) 5° F.

CHAPTER II

ADDITION, SUBTRACTION, AND PARENTHESES

ADDITION OF MONOMIALS

31. While in arithmetic the word *sum* refers only to the result obtained by adding positive numbers, in algebra this word includes also the results obtained by adding negative, or positive and negative numbers.

In arithmetic we add a gain of \$6 and a gain of \$4, but we cannot add a gain of \$6 and a loss of \$4. In algebra, however, we call the aggregate value of a gain of 6 and a loss of 4 the sum of the two. Thus a gain of \$2 is considered the sum of a gain of \$6 and a loss of \$4. Or in the symbols of algebra

$$(+ \$6) + (- \$4) = + \$2.$$

Similarly, the fact that a loss of \$6 and a gain of \$4 equals a loss of \$2, may be represented thus

$$(- \$6) + (+ \$4) = (- \$2).$$

In a corresponding manner we have for a loss of \$6 and a loss of \$4

$$(- \$6) + (- \$4) = (- \$10).$$

Since similar operations with different units always produce analogous results, we *define* the sum of two numbers in such a way that these results become general, or that

$$6 + (-4) = +2,$$

$$(-6) + (+4) = -2,$$

$$(-6) + (-4) = -10,$$

and

$$(+6) + (+4) = +10.$$

32. These considerations lead to the definition of addition:

If two numbers have the same sign, add their absolute values; if they have opposite signs, subtract their absolute values and (always) prefix the sign of the greater.

EXERCISE 10

Find the sum of:

1. $\begin{array}{r} +6 \\ -3 \\ \hline \end{array}$	2. $\begin{array}{r} -6 \\ +3 \\ \hline \end{array}$	3. $\begin{array}{r} -3 \\ -6 \\ \hline \end{array}$	4. $\begin{array}{r} +3 \\ +6 \\ \hline \end{array}$
5. $\begin{array}{r} -7 \\ +8 \\ \hline \end{array}$	6. $\begin{array}{r} +7 \\ -8 \\ \hline \end{array}$	7. $\begin{array}{r} 11 \\ -4 \\ \hline \end{array}$	8. $\begin{array}{r} -11 \\ -4 \\ \hline \end{array}$
9. $\begin{array}{r} 9 \\ -9 \\ \hline \end{array}$	10. $\begin{array}{r} -20 \\ -12 \\ \hline \end{array}$	11. $\begin{array}{r} 0 \\ 7 \\ \hline \end{array}$	12. $\begin{array}{r} 0 \\ -7 \\ \hline \end{array}$
13. $\begin{array}{r} 6 \\ 6 \\ -11 \\ \hline \end{array}$	14. $\begin{array}{r} -1 \\ -2 \\ -3 \\ \hline \end{array}$	15. $\begin{array}{r} 1 \\ -2 \\ +4 \\ \hline \end{array}$	16. $\begin{array}{r} -17 \\ 12 \\ +5 \\ \hline \end{array}$

Find the value of:

- | | |
|-----------------------|-----------------------------------|
| 17. $(-20) + (-21)$. | 20. $0 + (-18)$. |
| 18. $(-12) + 13$. | 21. $(-1) + 2 + (-3) + 4$. |
| 19. $1 + (-7)$. | 22. $(-6) + (-7) + (-8) + (-9)$. |

In Exs. 23-26, find the numerical value of $a + b + c + d$, if:

23. $a = 1, b = -2, c = -4, d = 6$.

24. $a = 7, b = -7, c = 0, d = 1$.

25. $a = -7, b = 8, c = -9, d = 10$.

26. $a = 17, b = 12, c = -17, d = -11$.

27. What number must be added to 3 to give 7?

28. What number must be added to 7 to give 3?

29. What number must be added to -7 to give 3 ?
30. What number must be added to -3 to give -7 ?
31. Add 4 yards, 7 yards, and 3 yards.
32. Add $4a$, $7a$, and $3a$.
33. Add $4a^2bx$, $7a^2bx$, and $-3a^2bx$.

33. Similar or like terms are terms which have the same literal factors, affected by the same exponents.

$6ax^2y$ and $-7ax^2y$, or $-5a^2b$ and $\frac{a^2b}{7}$, or $16\sqrt{a+b}$ and $-2\sqrt{a+b}$, are similar terms.

Dissimilar or unlike terms are terms which are not similar.

$4a^2bc$ and $-4a^2bc^2$ are dissimilar terms.

34. The sum of two similar terms is another similar term.

The sum of $3x^2$ and $-\frac{1}{2}x^2$ is $\frac{5}{2}x^2$.

Dissimilar terms cannot be united into a single term. The sum of two such terms can only be indicated by connecting them with the $+$ sign.

The sum of a and a^2 is $a + a^2$.

The sum of a and $-b$ is $a + (-b)$, or $a - b$.

35. Algebraic sum. In algebra the word **sum** is used in a wider sense than in arithmetic. While in arithmetic $a - b$ denotes a difference only, in algebra it may be considered either the difference of a and b or the sum of a and $-b$.

The sum of $-a$, $-2ab$, and $4ac^2$ is $-a - 2ab + 4ac^2$.

EXERCISE 11

Add:

- | | | | |
|-------------------------|---------------------------|---------------------------|----------------------------|
| 1. $-3a$ | 2. a^2 | 3. $7x^2y$ | 4. $-4mn^3$ |
| $-4a$ | $-2a^2$ | $-7x^2y$ | $-5mn^3$ |
| <u>$-5a$</u> | <u>$+3a^2$</u> | <u>$+x^2y$</u> | <u>$-7mn^3$</u> |

5. $12\ mxz$	7. $7\ mnp$	9. $14\ abx^3$
$15\ mxz$	$-8\ mnp$	$15\ abx^3$
$-28\ mxz$	$-9\ mnp$	$-17\ abx^3$
<hr/>	<hr/>	<hr/>
6. c^2de	8. $8\ a^2bc^2$	10. $11\ a^2b^2$
$-c^2de$	$-9\ a^2bc^2$	$-21\ a^2b^2$
$4\ c^2de$	$11\ a^2bc^2$	$+11\ a^2b^2$
<hr/>	<hr/>	<hr/>

Find the sum of:

11. $6\ a, 7\ a, -2\ a, -12\ a, -15\ a.$
12. $-7\ xy, 12\ xy, -19\ xy, -7\ xy.$
13. $-8\ abc^3, -7\ abc^3, -9\ abc^3, -abc^3.$
14. $4(a+b), -5(a+b), 6(a+b), -7(a+b).$
15. $-6\sqrt{x+y}, -7\sqrt{x+y}, -8\sqrt{x+y}, -9\sqrt{x+y}.$
16. $4\sqrt{a+b+c^3}, -8\sqrt{a+b+c^3}, -7\sqrt{a+b+c^3}.$

Find the value of:

17. $-11\ xy - 19\ xy - xy - 27\ xy - 30\ xy.$
18. $15\ m + 18\ m + m - 7\ m - 10\ m - 14\ m.$
19. $-\frac{11}{2}z^2 - \frac{8}{3}z^2 - \frac{13}{4}z^2 + \frac{4}{3}z^2.$
20. $5\ mn - 6\ mn + 7\ mn - 9\ mn + 12\ mn.$
21. $\frac{1}{3}ab^2 - \frac{2}{3}ab^2 + \frac{4}{3}ab^2 + \frac{3}{4}ab^2.$
22. $1.2\ x^2yz + 2.4\ x^2yz + .7\ x^2yz - 1.3\ x^2yz.$
23. $\frac{3}{2}p^2q + \frac{3}{5}p^2q - \frac{2}{5}p^2q.$
24. $\frac{1}{2}x^2y^2z^2 + \frac{1}{4}x^2y^2z^2 - \frac{3}{4}x^2y^2z^2 - x^2y^2z^2.$

Add:

25. a	26. a	27. $-a$	28. a
b	$-b$	$-b$	$-a^2$
<hr/>	<hr/>	<hr/>	<hr/>
29. a	30. a^2	31. a^3	32. -2
1	-1	$-b$	b
<hr/>	<hr/>	<hr/>	<hr/>

33. $\begin{array}{r} -ax \\ -ax^2 \\ \hline \end{array}$	34. $\begin{array}{r} 4abc \\ 4ab^2c \\ \hline \end{array}$	35. $\begin{array}{r} ab \\ 0 \\ \hline \end{array}$	36. $\begin{array}{r} \sqrt{a+b} \\ -\sqrt{a-b} \\ \hline \end{array}$
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Simplify the following by uniting like terms:

37. $6a + 4b - 5a - 15b + 2a + 4a - 7b + b.$

38. $10c - 11m + 5x - 4y - 4x - 12c + y + x - m.$

39. $12mn^3 - 12mn^2 - 11mn^3 + 11mn^2 - 17x.$

40. $\sqrt{a+b} + \sqrt{a+b^2} - 2\sqrt{a+b} - 3\sqrt{a+b^2} + \sqrt{a+b^2}.$

41. $6x^2y - 17x^2y - 19mn + 12mn - 6mn - x^2y.$

ADDITION OF POLYNOMIALS

36. Polynomials are added by uniting their like terms. It is convenient to arrange the expressions so that like terms may be in the same vertical column, and to add each column.

Thus, to add $26ab - 8abc - 15bc$, $-12ab + 15abc - 20c^2$, $-5ab + 10bc - 6c^2$, and $-7abc + 4bc + c^2$, we proceed as follows:

$$\begin{array}{r}
 26ab - 8abc - 15bc \\
 -12ab + 15abc - 20c^2 \\
 -5ab + 10bc - 6c^2 \\
 -7abc + 4bc + c^2 \\
 \hline
 9ab - bc - 25c^2 \quad \text{Sum.}
 \end{array}$$

37. Numerical substitution offers a convenient method for checking the sum of an addition. To check the addition of $-3a + 4b + 5c$ and $+2a - 2b - c$ assign any convenient numerical values to a , b , and c , e.g. $a = 1$, $b = 2$, $c = 1$,

then $-3a + 4b + 5c = -3 + 8 + 5 = 10,$

$$+2a - 2b - c = 2 - 4 - 1 = -3,$$

the sum $-a + 2b + 4c = -1 + 4 + 4 = 7.$

But $7 = 10 - 3$, therefore the answer is correct.

NOTE. While the check is almost certain to show any error, it is not an absolute test, e.g. the erroneous answer $-a + 6b - 4c$ would also equal 7.

38. In various operations with polynomials containing terms with different powers of the same letter, it is convenient to **arrange the terms** according to ascending or descending powers of that letter.

$7 + x + 5x^2 + 7x^3 + 5x^5$ is arranged according to ascending powers of x . $5a^7 - 7a^6b + 4a^4bc - 8a^2bd^2 + 7ab^4d + 9b^5 + e^7$ is arranged according to descending powers of a .

EXERCISE 12

Add the following polynomials :

1. $5a - 6b - 7c$, $-3a + 2b - 9c$, and $-8a - 5b + 11c$.
2. $8x - 5y + 7z$, $5x + 9y - 8z$, $-4x - 5y + 3z$, and $-14x + 6y - z$.
3. $2a^2 - 9b^2 - 3c^2$, $-5a^2 + 11b^2 - 9c^2$, $4a^2 - 3b^2 + 5c^2$, and $-6b^2 - 7c^2 + 8a^2$.
4. $5r - 6x - 9z + 11v$, $7r - 9x - 11z + 8v$, $4r - 8z$, and $8x + 14v$.
5. $26m + 10x + 14v + z$, $-12m + 15x - 20z$, $-12x - 5m - 5z$, and $11z - 7x$.
6. $12m - 14p + 13z$, $\div 4m + 3p + y$, $7m - x - 5y$, $-8m + 2x - y$, and $-10m - 8p + 4y$.
7. $-5xy - 2yz - 7xz$, $8xy + 3yz - 2z^2$, $-2xy + 4xz + 5z^2$, $-xy + yz - 4z^2$.
8. $3a^2 - 2b^2 + c^2$, $-2a^2 + b^2 - 3c^2$, $-a^2 + 3b^2 - 2c^2$, $a^2 + 2b^2 + 4c^2$.
9. $13(a + b) - 5(b + c) + 7(c + d)$, $5(a + b) + 9(b + c) - 8(c + d)$, $-4(a + b) - 5(b + c) + 3(c + d)$, and $-14(a + b) - (c + d) + 6(b + c)$.
10. $4\sqrt{x} - \sqrt{y} - 3\sqrt{z}$, $2\sqrt{y} + \sqrt{z}$, and $-4\sqrt{x} - \sqrt{y} - \sqrt{z}$.
11. $a^5 - a^4 + 2a^3$, $3a^5 - 4a^4 + 6a^3$, $-8a^5 - 7a^4 + 8a^3$, and $-3a^5 - 9a^4 - 16a^3$.

12. $1 + x - x^2$, $1 - x + x^2$, $-1 + x + x^2$, and $-x + x^4$.
13. $8(x + y)^2 - 7(x + y) + 6$, $-5(x + y)^2 - 3(x + y) - 5$, and $3(x + y)^2 + 9(x + y) + 2$.
14. $a^3 + 3a^2b + 3ab^2 + b^3$, $-2a^3 - 2b^3$, $a^3 - 2a^2b - b^3$, and $-2a^2b - b^3$.
15. $21pq - 17xy + 9y^2$, $-21pq - 9y^2 + xy$, and $y^2 - pq + 17xy$.
16. $a - b$, $b - c$, $c - d$, $d - e$, and $e - a$.
17. $7a^3 + 4b^3 - c^3 + d^3$, $c^3 - d^3 - e^3$, $b^3 - a^3 + c^3$, and $c^3 - b^3 - a^3$.
18. $-3.5a - 5.7b + 1.8c$, $5.3a - 4.3c - 3.6b$, $11.2b - 2.2c - 7.4a$.
19. $x^4 - 2x^2 + x^3 - 1$, $2x^3 - 2x^4 + 1$, $2 + x^2$.
20. $x^3 - 2y^3 - 11x^2y - xy^2$, $4y^3 - 3x^3 + 2x^2y$, $7x^2y - 6xy^2 + x^3$, $7x^3 - 4xy^2 + 4x^2y$.
21. $2a^3 - a^2 - a$, $4a^3 - 6a^2 + a$, $-a^3 + 8a^2 + 7a$.
22. $3m^3 + 5m + 8$, $10m^3 - 6 - 4m^2$, $2m^3 - 2m - 3$.
23. $4p^3 + 7p^4 - p + 1$, $6p^2 - 3p^3 + p$, $7 - 2p^4 + p^2$, $-p^3 + 4p - p^2$.
24. $b^4 - b^3 + b^2 - b + 1$, $-2b^4 + 2b^3 - 2b^2 - 2b + 2$, $3b^4 - 3b^3 + 3b^2 - 3b + 3$.
25. $-9a + 3 + 16a^3 + a^2$, $13a^2 + 5 - 4a + 8a^3$, $11a - 15 + 7a^2 + 6a^3$.
26. $5a^3 - 4a^2b + 3ab^2 - 2b^3$, $-4a^3 + 3a^2b - 2ab^2 + b^3$, $3a^3 - 2a^2b + ab^2$, $4a^3 + 3a^2b - 2ab^2 + b^3$.
27. $7a^4 - 4b^4 + 3a^3b - 2a^2b^2 + 7ab^3$, $-7ab^3 + 4a^3b - 7a^4 + 3a^2b^2$, $b^4 - 3a^2b^2$.
28. $6a^4 - 7b^4$, $3a^3b + 3a^2b^2$, $6a^2b^2 - 5a^4$, $b^4 - 6a^3b$.
29. $\frac{1}{2}x^2 - 2x^3 + \frac{1}{3}x - 3$, $\frac{1}{2}x^3 - \frac{3}{2}x^2 + \frac{1}{2}$, $2x - \frac{3}{2}x^2$.
30. $\frac{1}{2}x^2 - \frac{1}{3}x - \frac{1}{4}$, $\frac{1}{3}x^2 - \frac{1}{2}x - \frac{1}{4}$, $\frac{1}{6}x^2 - \frac{1}{6}x - \frac{1}{2}$.

SUBTRACTION

EXERCISE 13

1. What is the remainder if 6 is taken from 12?
 2. If from the 6 negative units, $-1, -1, -1, -1, -1, -1$, four negative units are taken, how many negative units remain? What is therefore the remainder when -4 is taken from -6 ?
 3. Instead of subtracting in the preceding example, what number may be added to obtain the same result?
 4. The sum total of the units $+1, +1, +1, +1, +1, -1, -1$, and -1 , is 2. What is the value of the sum if two negative units are taken away? If three negative units are taken away?
 5. What is therefore the remainder when -2 is taken from 2? When -3 is taken from 2?
 6. What other operations produce the same result as the subtraction of a negative number?
 7. If you diminish a person's debts, does he thereby become richer or poorer?
 8. State other practical examples which show that the subtraction of a negative number is equal to the addition of a positive number.
-

39. Subtraction is the inverse of addition. In addition, two numbers are given, and their algebraic sum is required. In subtraction, the algebraic sum and one of the two numbers is given, the other number is required. The algebraic sum is called the *minuend*, the given number the *subtrahend*, and the required number the *difference*.

Therefore any example in subtraction may be stated in a different form; *e.g.* from -5 take -3 , may be stated: What number added to -3 will give -5 ? To subtract from a the number b means to find the number which added to b gives a .

Or in symbols,
 $a - b = x,$
 if $x + b = a.$

Ex. 1. From 5 subtract -3 .

The number which added to -3 gives 5 is evidently 8.

Hence, $5 - (-3) = 8.$

Ex. 2. From -5 subtract -3 .

The number which added to -3 gives -5 is -2 .

Hence, $(-5) - (-3) = -2.$

Ex. 3. From -5 subtract $+3$.

This gives by the same method,

$$-5 - (+3) = -8.$$

40. The results of the preceding examples could be obtained by the following

Principle. *To subtract, change the sign of the subtrahend and add.*

The numerical results of Exs. 1-3 of course do not prove this principle, but it may be deduced as follows:

The principle is obviously correct for a positive subtrahend.

To find $a - (-b)$ we have to find the number which added to $-b$ will give the result a .

But $a + b$ added to $-b$, gives a .

Hence the required remainder is $a + b$,

or, $a - (-b) = a + b.$

NOTE. The student should perform *mentally* the operation of changing the sign of the subtrahend; thus to subtract $-8a^2b$ from $-6a^2b$, change mentally the sign of $-8a^2b$ and find the sum of $-6a^2b$ and $+8a^2b$.

41. To subtract polynomials we change the sign of each term of the subtrahend and add.

Ex. From $-6x^3 - 3x^2 + 7$ subtract $2x^3 - 3x^2 - 5x + 8$.

$$\begin{array}{r}
 -6x^3 - 3x^2 + 7 \\
 2x^3 - 3x^2 - 5x + 8 \\
 \hline
 -8x^3 + 5x - 1
 \end{array}
 \qquad
 \begin{array}{l}
 \text{Check, If } x = 1 \\
 = -2 \\
 = +2 \\
 \hline
 = -4
 \end{array}$$

EXERCISE 14

From

1. 6 take 8.
2. -8 take 6.
3. 8 take -6 .
4. -8 take -8 .
5. -6 take -8 .
6. 6 take -8 .
7. 11 take -11 .
8. -11 take $+11$.
9. -11 take -11 .
10. 13 take -17 .
11. -12 take 7.
12. 0 take 64.

Subtract the following numbers :

13. $\begin{array}{r} 11x^3 \\ 11x^3 \\ \hline \end{array}$
14. $\begin{array}{r} 12x^2y \\ -12x^2y \\ \hline \end{array}$
15. $\begin{array}{r} -x^3y \\ -2x^3y \\ \hline \end{array}$
16. $\begin{array}{r} a^2bc^2 \\ 2a^2bc^2 \\ \hline \end{array}$
17. $\begin{array}{r} 36mnp \\ -22mnp \\ \hline \end{array}$
18. $\begin{array}{r} 0 \\ -12xyz \\ \hline \end{array}$
19. $\begin{array}{r} 17ad \\ 21ad \\ \hline \end{array}$
20. $\begin{array}{r} -41a^2b^2 \\ +41a^2b^2 \\ \hline \end{array}$

21. Subtract $17xyz$ from $6xyz$.

✓ 22. Subtract the sum of $6x$ and $7x$ from $-12x$.

23. From $26a + 38b + 12c$ subtract $26a - 14b + 18c$, and check the answer.

24. From $7a^2 + 12a + 3$ subtract $2a^2 + 13a - 7$, and check the answer.

25. From $x^2 - 7xy + y^2$ take $x^2 - 7xy - y^2$.

26. From $-81a^2 + 12ab - 7b^2$ subtract $81a^2 + 22ab - 14b^2$.

27. From $6a - 7b + 8c - 9d$ take $12a - 7b - 12c + 17d$.
28. From $6x^3 - 17x^2 + 5x - 3$, subtract $3 - 7x + 5x^2 - 9x^3$.
29. From $7x^4 - 2x^3 + 9$ subtract $4 - 7x + 3x^2 - 9x^4$.
30. From $a^3 - 3a^2b + 3ab^2 - b^3$ take $-a^3 + 3a^2b - 3ab^2 + b^3$.
31. From $ab - bc + cd - da$, subtract $ab + bc + cd + da$.
32. From $-2 + 8x^3 - 4x^2 - 2x$ take $2x^3 - 7x^2 + 3x$.
33. From $m^2 - 3mn + 6n^2$ take $6n^2 - m^2 + 7mn$.
- ✓ 34. From $x + y$ take $x + y + z + u$.
35. From $19ab + 17bc - 14ac$ take $17ab + 19bc + 14ac$.
- ✓ 36. Subtract $x + y$ from $x - y + z - u$.
37. Subtract $7x^3 - 6x^2 - 5x + 2$ from $7 - 6x - 5x^2 + 2x^3$.
38. Subtract $2ax - 7by - 5xy$ from $2by - 2ax - 5xy$.
39. From $2x + y$ take $x + y + z$.
40. From $x^5 + 1$ take $-x^4 - x^3 - x^2 - x$.
41. From $-x^4 - x^2 - x$ subtract $x^5 - 1$.
42. Subtract $a + b + c$ from d .
43. From $x + \sqrt{x} - \sqrt{y}$ take $2x - 2\sqrt{x} - 2\sqrt{y}$.
44. Subtract $\sqrt{ax} + \sqrt{bx} - \sqrt{cx}$ from $\sqrt{ax} + \sqrt{bx} - \sqrt{cx}$.
45. From $6(x + y) + 4\sqrt{x} - \sqrt{x + y}$ subtract $7(x + y) - 5\sqrt{x} + 7\sqrt{x + y}$.
46. Subtract $x^3 + \frac{1}{2}x^2 - \frac{1}{4}x - \frac{3}{4}$ from $2x^3 - \frac{1}{2}x^2 + \frac{1}{4}x - \frac{5}{4}$.
47. Subtract $\frac{1}{5}a^3 - \frac{2}{5}a^2 - \frac{3}{5}a - \frac{4}{5}$ from $\frac{2}{5}a^3 + \frac{3}{5}a^2 - \frac{4}{5}a + 1$.

REVIEW EXERCISE I

1. To the sum of $3a - 4b + c$ and $6a - 2b - 7c$ add the sum of $-7a - 6b - c$ and $-a + b - 8c$.
2. From the sum of $a - 2b + 3c$ and $2a - 3b + 4c$ subtract $4a - 5b + 6c$.
3. From $x + y$ subtract the sum of $2x - y + z$ and $x - 3y + 4z$.

4. From the difference of $x^2 - x$ and $x - 1$ subtract $x^2 + x + 1$.
- (5.) Subtract the sum of $5x^2 - 6x + 7$ and $-4x^2 - 6x + 10$ from $x^2 - 6x + 16$.
6. Subtract the difference of $4m^2 - 2m - 7$ and $3m^2 - 2m + 7$ from unity.
7. Subtract the difference of $6a^2 - 7a + 2$ and $-4a^2 + 5a + 2$ from zero.
- (8.) Subtract the sum of $a + b$ and $a - b$ from $a + b + c$.
9. Subtract the sum of $x + y + z$ and $x - y - z$ from the difference of $x - y + z$ and $-x + y + z$.
10. What expression must be added to $6a + 2b$ to produce $7a - 3b$?
11. What expression must be added to $a^2 - a$ to produce $a + b$?
12. What expression must be subtracted from $3a + 2b + c$ to produce $2a - b - c$?
13. From what expression must $a + b$ be subtracted to produce a difference of $2a - 3b + c$?
14. What must be added to $-3a + 2b - c$ to produce zero?
15. What must be subtracted from $a - b$ to produce zero?
- (16.) By how much does $a + b + c$ exceed $a - b + c$?
17. What number is less than $3a + 2b$ by $-3a + 2b$?
- ✓ 18. Add 3×2^a , -5×2^a , and -9×2^a .
- (19.) Represent the sum of $6 \times 2^3 - 7 \times 2^3 + 2^3 + 3 \times 2^3$ as a multiple of 2^3 .
20. Add $135 \times 3^4 - 140 \times 3^4 + 20 \times 3^4 - 24 \times 3^4$.
- (21.) A is a years old. How old will he be 10 years hence? a years hence?
22. A is 10 years old. How old will he be in $a + b$ years? In $2a - 10$ years?
- (23.) A was n years old when B was born. How old will A be when B is $2n$ years old?

If $a + b + c = m$, $a + b - c = n$, $2a - 3b + 4c = p$, find

24. $m + n$.

26. $m + n + p$.

28. $m + n - p$.

25. $m - n$.

27. $m - n + p$.

29. What is the remainder obtained by subtracting any number from another one which is smaller by 10?

30. If $6a + 2b$ is the subtrahend and $2a - 3b$ the remainder, find the minuend.

31. Can you discover a short method for adding the ten consecutive numbers, 6 7 3 2 4 0, 6 7 3 2 4 1, 6 7 3 2 4 2, and so on to 6 7 3 2 4 9?

HINT. Consider each number a binomial.

32. If $3 \cdot 2^4 - 5 \cdot 2^4 + 7 \cdot 2^4 = 2^4 \cdot x$, find x .

SIGNS OF AGGREGATION

42. By using the signs of aggregation, additions and subtractions may be written as follows:

$$a + (+b - c + d) = a + b - c + d.$$

$$a - (+b - c + d) = a - b + c - d.$$

Hence it is obvious that parentheses preceded by the + or the - sign may be removed or inserted according to the following principles:

43. I. *A sign of aggregation preceded by the sign + may be removed or inserted without changing the sign of any term.*

II. *A sign of aggregation preceded by the sign - may be removed or inserted provided the sign of every term inclosed is changed.*

$$\text{E.g. } a + (b - c) = a + b - c.$$

$$a + (-b + c) = a - b + c.$$

$$a - (+b - c) = a - b + c.$$

$$a - -b - c = a + b + c.$$

44. If there is no sign before the first term within a parenthesis, the sign $+$ is understood.

$$a - (b - c) = a - b + c.$$

45. If we wish to remove several signs of aggregation, one occurring within the other, we may begin either at the innermost or outermost. The beginner will find it most convenient at every step to remove only those parentheses which contain no others.

Ex. Simplify $4a - \{7a + 5b - [-6b + (-2b - \overline{a - b})]\}$.

$$\begin{aligned} & 4a - \{7a + 5b - [-6b + (-2b - \overline{a - b})]\} \\ &= 4a - \{7a + 5b - [-6b + (-2b - a + b)]\} \\ &= 4a - \{7a + 5b - [-6b - 2b - a + b]\} \\ &= 4a - \{7a + 5b + 6b + 2b + a - b\} \\ &= 4a - 7a - 5b - 6b - 2b - a + b \\ &= -4a - 12b. \quad \text{Answer.} \end{aligned}$$

EXERCISE 15

Simplify the following expressions:

1. $a + (b - 2c).$
2. $a - (b - 2a).$
3. $a^2 - (2b^2 + a^2 - c^2).$
4. $a^3 - (+a^3 - b^3).$
- ✓ 5. $a - (-a - 2b + c) + (-a + b).$
- ✓ 6. $2x - \overline{x - y} + (-x + y).$
- ✓ 7. $4x + \overline{+3x + y}.$
- ✓ 8. $6x^2 + 94x - 7 - (5x^2 + 3x) + 6.$
9. $(6a + 6b) - (5a - 4b) + (3a - 2b).$
- ✓ 10. $17x - (16x + y) + (4x - y + z) - \overline{x + z}.$
- ✓ 11. $(a + b + c) - (a + b - c) - (a - b + c).$

12. $a + [b - (a - b)]$.
13. $a + b - [(b + d) - (a - b)]$.
14. $m - (n - p) + [3m - \overline{3n - 6m}]$.
15. $a - [a - \{a - (-a)\}]$.
16. $x - [3y + \{3z - (z - x) + y\} - 2x]$.
17. $-[m - (m + n) - (m - n) - (-m + n)]$.
18. $12a - \{(a + b) - [b - (a - b)] - a\}$.
19. $2x - \{x - (x - y) - [x - \overline{x - y}] - y\}$.
20. $12 - 2a - \{-a - [2a - (a - \overline{7 - a})]\}$.
21. $14 - 3a - \{9a - [10a - (11a - \overline{6 - 6a})]\}$.
22. $a - [-\{-(-a)\}]$.
23. $a - 3 - [-\{-(-a + \overline{a + b})\}]$.
24. $x + y - [-(x - y) + \{-x + \overline{(x - y)}\}]$.
25. $1 - \{-a - (a + 1) - [-a - (a - \overline{a - 1})]\}$.
26. $a - (-\{-[-(-a)]\})$.
27. $1 - (-\{a + (-a + 1)\}) - \{a - \overline{a - 1}\}$.
28. $6m + \{4m - [8n - (2m + 4n) - 22n] - 7n\}$
 $+ [9m - (3n + 4m) + 14n]$.
29. $1247 - [1722 - \{1722 + (933 - 1247)\}]$.
30. From $a + \{(4 - b) + (a - 4) - \overline{a - 7}\}$ subtract
 $a - \{(6 - b) + (6a - 6) - (5a - 7)\}$.
31. From the sum of $a + \{a - (b - c)\}$ and
 $-a + [4a - (5b + c)]$ subtract $a - (b - c)$.
32. Simplify $4a - [6b + (3a - c) - \{5b - \overline{c - a}\}]$ and check the answer by substituting $a = 3, b = 2, c = 1$ in the question and the answer.
33. Simplify $9a - [-7a + \{5b - (a - b) + \overline{a - b}\}]$ and check the answer by the substitution $a = 1, b = 2$.

46. Signs of aggregation may be inserted according to § 43.

Ex. 1. In the following expression inclose the second and third and the fourth and fifth terms respectively in parentheses:

$$\begin{aligned} & a - b + c + 2d - e \\ &= a - (b - c) + (2d - e). \end{aligned}$$

Ex. 2. Inclose in a parenthesis preceded by the sign $-$ the last three terms of

$$\begin{aligned} & 2a + b - 5c + 2d \\ &= 2a - (-b + 5c - 2d). \end{aligned}$$

EXERCISE 16

In each of the following expressions inclose the last three terms in a parenthesis:

- | | |
|------------------------|-------------------------------|
| 1. $e + b + c - d.$ | 3. $2x - 6x^2 - 7x^3 + 5x^4.$ |
| 2. $x - 2y + 3z - 4d.$ | 4. $2p - q + p^2 - q^2.$ |

In each of the following expressions inclose the last three terms in a parenthesis preceded by the minus sign:

- | | |
|-----------------------|--------------------------------|
| ✓ 5. $a - b - c - d.$ | 8. $a + a^2 - b + b^2.$ |
| 6. $a - b + c + d.$ | ✓ 9. $1 + a - 2b - c - d.$ |
| 7. $a + 2b - 5c - d.$ | ✓ 10. $x - y + z + m + n + p.$ |

EXERCISES IN ALGEBRAIC EXPRESSION

EXERCISE 17

Write the following expressions:

1. The sum of the squares of a and b .
2. The square of the sum of a and b .
3. The difference of the cubes of x and y .

NOTE. The minuend is always the first, and the subtrahend the second, of the two numbers mentioned.

4. The difference of the cubes of y and x .
5. The cube of the difference of x and y .
6. The product of a and b .
7. The product of the cubes of a and b .
8. The cube of the product of a and b .
9. The product of the sum and the difference of a and b .
10. Six times the square of the difference of a and b diminished by the quantity a minus b .
11. The product of the difference of a and b and the quantity $a^2 - 2b^2 + c^3$.
12. The sum of a and b diminished by the difference of a and b .
13. a cube minus the quantity $2x^2$ minus $6y^2$ plus $7c^2$ plus the quantity $-x + y$.
14. The sum of the cubes of a , $-b$, and c divided by the difference of a and c .

Write algebraically the following statements:

15. The sum of a and b multiplied by the difference of a and b is equal to the difference of a^2 and b^2 .
16. The difference of the cubes of a and b divided by the difference of a and b is equal to the square of a plus the product of a and b , plus the square of b .
17. The difference of the squares of two numbers divided by the difference of the numbers is equal to the sum of the two numbers. (Let a and b represent the numbers.)

CHAPTER III

MULTIPLICATION

MULTIPLICATION OF ALGEBRAIC NUMBERS

EXERCISE 18

1. If a man makes \$15 a day, how many dollars will he make in 5 days?
2. If a man loses \$15 a day, how many dollars will he make in terms of algebra in 5 days? (Denote gain by +, and loss by -.)
3. If from a man's fortune \$15 are deducted 5 times, how much in terms of algebra does he make?
4. If from a man's debts \$15 are deducted 5 times, how much does he gain?
5. Express each of the Exs. 1-4 as a multiplication example, considering gain as positive, and loss or debts as negative.
6. If we denote three days hence by +3, by what must we denote three days ago?
7. If we denote northerly motion as positive, and three days hence as +3, express the following as multiplication examples with algebraic symbols:
 - (a) A ship sailing north at the rate of 3° per day and crossing the equator to-day will, in 6 days, be 18° north of the equator.
 - (b) The same ship 6 days ago was 18° south of the equator.
 - (c) A ship sailing south at the rate of 3° per day and crossing the equator to-day will, in 6 days, be 18° south of the equator.
 - (d) The same ship 6 days ago was 18° north of the equator.

8. If the signs obtained for the products in the preceding examples were generally correct, what would be the value of

$$6 \times 3, 6 \times (-3), (-6) \times 3, (-6) \times (-3)?$$

9. State a rule by which the sign of the product of two numbers can be obtained.

47. Multiplication by a positive integer is a repeated addition; thus, 4 multiplied by 3, or $4 \times 3 = 4 + 4 + 4 = 12$,

-4 multiplied by 3, or $(-4) \times 3 = (-4) + (-4) + (-4) = -12$.

The preceding definition, however, becomes meaningless if the multiplier is a fractional or negative number. To take a number $2\frac{2}{3}$ or -7 times is just as meaningless as to go to bed $2\frac{2}{3}$ times or to fire a gun -7 times.

A more useful definition of multiplication which may be used in nearly all multiplication problems, is the following:

48. Multiplication is the operation of finding a number that has the same relation to one factor (multiplicand) as the other factor (multiplier) has to 1.

Thus $\frac{3}{5}$ is obtained from 1, by taking one fifth of unity three times, or

$$\frac{3}{5} = \frac{1}{5} + \frac{1}{5} + \frac{1}{5}.$$

Therefore 6 multiplied by $\frac{3}{5}$ is obtained by taking one fifth of 6 three times, or

$$6 \times \frac{3}{5} = \frac{6}{5} + \frac{6}{5} + \frac{6}{5}.$$

49. The product of 4 multiplied by -3 is obtained from 4 in the same manner in which -3 is obtained from 1.

But $-3 = -1 - 1 - 1$, i.e. -3 is obtained by subtracting three times 1.

Therefore, $(+4) \times (-3) = -(+4) - (+4) - (+4) = -12$.

and $(-4) \times (-3) = -(-4) - (-4) - (-4) = +12$.

In multiplying algebraic numbers we have therefore the following cases:

$$4 \times 3 = 12.$$

$$(-4) \times 3 = -12. \quad (\S 47.)$$

$$4 \times (-3) = -12.$$

$$(-4) \times (-3) = +12.$$

50. Evidently the particular numbers chosen do not affect the sign of the answer, and hence the preceding illustrations lead to the

✓ **Law of Signs:** *The product of two numbers with like signs is positive; the product of two numbers with unlike signs is negative.*

To obtain the absolute value of a product we multiply the absolute values of the factors.

Thus $(+a)(+b) = +(ab).$

$$(+a)(-b) = -(ab).$$

$$(-a)(+b) = -(ab).$$

$$(-a)(-b) = +(ab).$$

51. Some fundamental laws which are correct for arithmetic, we shall assume for algebra.

✓ **I. Commutative Law.**

(a) For addition, $a + b = b + a,$

e.g. $4 + 5 = 5 + 4.$

(b) For multiplication, $ab = ba,$

e.g. $4 \cdot 5 = 5 \cdot 4.$

I.e. Algebraic numbers may be added (or multiplied) in any order.

✓ **II. Associative Law.**

(a) For addition,

$$a + b + c = a + (b + c) = (a + b) + c,$$

e.g. $4 + 5 + 6 = 4 + 11 = 9 + 6.$

(b) For multiplication,

$$abc = a(bc) = (ac)b = (ab)c.$$

$$\text{e.g. } 2 \cdot 3 \cdot 4 = 2 \cdot 12 = 8 \cdot 3 = 6 \cdot 4.$$

I.e. The sum (or the product) of three or more algebraic terms does not depend upon the grouping of the terms.

III. For the **Distributive Law** see § 55.

These laws may sometimes be used to shorten the arithmetical work,
e.g. $14 \cdot (-11) \cdot (\frac{1}{7}) \cdot (-\frac{2}{11}) = (14 \cdot \frac{1}{7}) \cdot (-11 \cdot -\frac{2}{11}) = 2 \times 2 = 4.$

EXERCISE 19

Find the values of the following products:

1. $6 \times (-3).$

4. $(-15) \times (-5).$

2. $(-7) \times (+7).$

5. $4 \times (-13).$

3. $(-22) \times (+7).$

6. $13 \times (-3).$

NOTE. If no misunderstanding is possible the parentheses about factors are frequently omitted.

7. $-16 \times -4.$

✓ 13. $(-2)^3.$

8. $+4 \cdot -4.$

✓ 14. $(-3)^4.$

9. $-7 \cdot (2\frac{1}{7}).$

15. $(-1)^{10}.$

10. $2\frac{1}{2} \cdot -\frac{5}{2}.$

✓ 16. $0^{10}.$

11. $-5 \cdot 27 \cdot 20.$

17. $(-10)^6.$

✓ 12. $(-10\frac{1}{9}) \cdot -\frac{3}{7} \cdot -9 \cdot +\frac{7}{3}.$

18. $(-\frac{1}{10})^3.$

19. Formulate a law of signs for a product containing an even number of negative factors.

20. Formulate a law of signs for a product containing an odd number of negative factors.

If $a = 3$, $b = -2$, $c = 1$, $d = 0$, and $x = 5$, find the numerical value of:

21. $6abc.$

23. $7ab^3c^9.$

22. $4a^2b^2.$

✓ 24. $-3a^3c^4.$

$$25. -7c^4dx.$$

$$26. 21ab^4c^{11}.$$

$$27. -(2abc)^2.$$

$$28. -2(abc)^2.$$

$$29. +2a(bc)^2.$$

$$30. -4b^x.$$

$$31. (-4c)^x.$$

$$32. 4a^2 - 5b^2 - 6c.$$

$$33. 4ab - 5b^3 + 6cd.$$

$$34. (a+b)^3 - c^3.$$

$$35. a + \{b^2 - (c+d)^3\}.$$

$$36. 7ab^2c^3 - 9bc^3 + 7c.$$

$$37. \sqrt{3a} + \sqrt{-2b} + \sqrt{d}.$$

$$38. 2^a - a^2.$$

MULTIPLICATION OF MONOMIALS

52. By definition, $a^3 = a \cdot a \cdot a$, and $a^5 = a \cdot a \cdot a \cdot a \cdot a$. Hence $a^3 \times a^5 = a \cdot a \cdot a \times a \cdot a \cdot a \cdot a \cdot a = a^8$, i.e. a^{3+5} . Or in general, if m and n are two positive integers,

$$\begin{aligned} a^m \times a^n &= (a \cdot a \cdot a \cdots \text{to } m \text{ factors}) \cdot (a \cdot a \cdot a \cdots \text{to } n \text{ factors}) \\ &= a \cdot a \cdots \text{to } (m+n) \text{ factors.} \end{aligned}$$

$$a^m \times a^n = a^{m+n}.$$

This is known as

The Exponent Law of multiplication: *The exponent of the product of several powers of the same base is equal to the sum of the exponents of the factors.*

53. To multiply two monomials, $5a^2b^3c$ and $-7a^3b^4d^4$, we apply the laws of association and commutation,

$$\begin{aligned} 5a^2b^3c \times -7a^3b^4d^4 &= (5 \cdot -7) \cdot (a^2 \cdot a^3) \cdot (b^3 \cdot b^4) \cdot c \cdot d^4 \\ &= -35a^5b^7cd^4. \end{aligned}$$

EXERCISE 20

Express each of the following products as a power:

$$1. a^3 \cdot a^2.$$

$$2. b^4 \cdot (-b^3).$$

$$3. (xy)^7 \cdot (xy)^6.$$

$$4. (x+y)^3 \cdot (x+y)^4.$$

$$5. 2^3 \cdot 2^7.$$

$$6. 143^{11} \cdot 143^{12}.$$

7. $a^{2n} \cdot a^{3n}$.

10. $x^6 \cdot x^2 \cdot x(-x^3)$.

8. $x^{a+b} \cdot x^{a-b}$.

11. $a^{n-1} \cdot a$.

9. $2^2 \cdot 2^3 \cdot 2^4 \cdot 2$.

12. $a^{n-2} \cdot a \cdot (-a)$.

Perform the multiplications indicated:

13. $3x \cdot 5xy$.

19. $7xyz \cdot 3mn$.

14. $3xy \cdot (-5xy^2)$.

20. $25x^2y^2z^2 \cdot (-8xyz)$.

15. $-7ab \cdot 3a^2b^2c^2$.

21. $(-6x^2y^2z^2) \cdot (-3xyz^4)$.

16. $22a^2b^3 \cdot (-5ab^5c)$.

22. $6ax^2y \cdot (-7by^2z)$.

17. $(-11a^4b^4c^9) \cdot (-27a^5b^5c)$.

23. $7a^3b^2c \cdot (9ab^2c^3)$.

18. $(-4a^4bc) \cdot 2a^3b^2cd$.

24. $(-12pqr) \cdot 12mnr$.

25. $3a^2b^2c^2 \cdot (-4abc) \cdot (-5a^2c^3)$.

26. $(-2a^5bx) \cdot (-2ab^5x) \cdot 2abx^5$.

27. $7mn \cdot (-5np) \cdot (-2pq)$.

28. $(-3a^3) \cdot (-2b^3) \cdot (-c)$.

29. $2(x+y)^2 \cdot 3(x+y) \cdot 3(x+y)$.

30. $3(a-b)^5 \cdot -2(a-b)^4 \cdot (a-b)^2$.

31. $6a^2(m+n) \cdot (-7ab)$.

32. $(-2a^nb^m) \cdot (a^nb^m)$.

34. $2^a \cdot 2^{2a} \cdot 2^{3a} \cdot 2^{4a}$.

33. $a^{n+1} \cdot a^{n-2} \cdot a$.

35. $(a+b+c) \cdot (a+b+c)^n$.

MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL

54. If we had to multiply 2 yards and 3 inches by 3, the results would evidently be 6 yards and 9 inches. Similarly the quadruple of $a + 2b$ would be $4a + 8b$, for

$$\begin{aligned} 4(a + 2b) &= (a + 2b) + (a + 2b) + (a + 2b) + (a + 2b) \\ &= 4a + 4(2b) = 4a + 8b. \end{aligned}$$

✓ 55. This principle, called the **distributive law**, is evidently correct for any positive integral multiplier, but we shall assume it for any number.

Thus we have in general

$$a(b + c) = ab + ac.$$

56. To multiply a polynomial by a monomial, *multiply each term by the monomial.*

$$-3a^2b(6a^2bc + 2bc - 1) = -18a^4b^2c - 6a^2b^2c + 3a^2b.$$

EXERCISE 21

Perform the multiplications indicated.

1. $3x(x + y + z).$
2. $-3xy(x^2 - y^2 - 1).$
3. $2mn(m^2 - n^2).$
4. $4a^2b^3c^2(-4a + 2abc - c^3).$
5. $-5a^2(-5a^2 - 5a - 5).$
6. $2xyz(7x^2y + 2yz^2 - 3x^2z).$
7. $-4xz(5x^2z - 3xz^2 + 2xz).$
8. $-2bc(a^4 - b^4 - 3c^4 + d^4).$
9. $-5x^4y(7xyz - 5xy^2z^2 - 2z^3).$
10. $6xy^2(2x^2 - 2xy - y^3 - 1).$
11. $4p^2q(-2pq + 2p^2 - 3q^2 + 1).$
- ✓ 12. $\frac{3}{2}a^2(4a^4 - 2ab - 4b^4).$
- ✓ 13. $-a^n(1 - a^n + a^{2n} - a^{3n}).$
- ✓ 14. $b^m(1 + b - b^n + b^m).$
- ✓ 15. $(a + b)^2[(a + b)^3 + (a + b)^2 + (a + b) + 1].$
- ✓ 16. By what expression must x be multiplied to give the product $ax - bx + cx$?
- ✓ 17. Express $3ax + 4bx - cx$ as a product.
18. How many x are contained in $mx - nx$?
19. Express the sum of $+n^2x$ and $-m^2x$ as a multiple of x .
20. Find the factors of $ab + ac - 3a$.
21. Find the factors of $3ab + 3bc - 3ac$.
- ✓ 22. Find the factors of $3x^3 - 6x^2 - 15x$.
23. Find the factors of $5a^2b - 15a^2b^2 - 60abc$.

MULTIPLICATION OF POLYNOMIALS

57. Any polynomial may be written as a monomial by inclosing it within a parenthesis. Thus to multiply $a - b$ by $x + y - z$, we write $(a - b)(x + y - z)$ and apply the distributive law.

$$\begin{aligned}(a - b)(x + y - z) &= x(a - b) + y(a - b) - z(a - b) \\ &= (ax - bx) + (ay - by) - (az - bz) \\ &= ax - bx + ay - by - az + bz.\end{aligned}$$

58. To multiply two polynomials, multiply each term of one by each term of the other and add the partial products thus formed.

The most convenient way of adding the partial products is to place similar terms in columns, as illustrated in the following example:

Ex. 1. Multiply $2a - 3b$ by $a - 5b$.

$$\begin{array}{r} 2a - 3b \\ \quad a - 5b \\ \hline 2a^2 - 3ab \\ \quad - 10ab + 15b^2 \\ \hline 2a^2 - 13ab + 15b^2 \end{array} \quad \text{Product.}$$

59. If the polynomials to be multiplied contain several powers of the same letter, the work becomes simpler and more symmetrical by arranging these expressions according to either ascending or descending powers.

Ex. 2. Multiply $2 + a^3 - a - 3a^2$ by $2a - a^2 + 1$.

Arranging according to ascending powers:

$2 - a - 3a^2 + a^3$	<i>Check.</i> If $a = 1$
$\quad 1 + 2a - a^2$	$= -1$
$\hline 2 - a - 3a^2 + a^3$	$= 2$
$\quad + 4a - 2a^2 - 6a^3 + 2a^4$	
$\quad \quad - 2a^2 + a^3 + 3a^4 - a^5$	
$\hline 2 + 3a - 7a^2 - 4a^3 + 5a^4 - a^5$	$\hline = -2$

60. Examples in multiplication can be checked by numerical substitution, 1 being the most convenient value to be substituted for all letters. Since all powers of 1 are 1, this method tests only the values of the coefficients and not the values of the exponents. Since errors, however, are far more likely to occur in the coefficients than anywhere else, the student should apply this test to every example.

Ex. 3. Multiply $3x^2y^4 + 6x^5y + y^6 - 2xy^5$ by $-2xy^3 + 4x^4 - 5y^4$.

Arranging according to descending powers of x :

Check. If $x = 1, y = 1$

$$\begin{array}{r}
 6x^5y + 3x^2y^4 - 2xy^5 + y^6 \\
 4x^4 - 2xy^3 - 5y^4 \\
 \hline
 24x^9y + 12x^3y^4 - 8x^5y^5 + 4x^4y^6 \\
 \quad - 12x^6y^4 \qquad \qquad - 6x^3y^7 + 4x^2y^8 - 2xy^9 \\
 \qquad \qquad - 30x^5y^5 \qquad \qquad - 15x^2y^8 + 10xy^9 - 5y^{10} \\
 \hline
 24x^9y \qquad - 38x^5y^5 + 4x^4y^6 - 6x^3y^7 - 11x^2y^8 + 8xy^9 - 5y^{10} = -24
 \end{array}$$

61. The degree of a term is equal to the number of literal factors contained in the term. Hence the degree of a term is also equal to the sum of the exponents of the literal factors.

$-17a^3b^2cd$ is a term of the 7th degree, since the sum of the exponents is $3 + 2 + 1 + 1$.

62. A homogeneous polynomial is a polynomial whose terms are all of the same degree; as $6x^3yz - 3xy^2z^2 - 2z^5$.

The product of two homogeneous polynomials must be homogeneous, since the degree of every term in the product must be equal to the sum of the degrees of the factors. This fact may be used as an additional check in some examples.

EXERCISE 22

Perform the following multiplications and check the results:

1. $(2a - 5b)(7a + 4b)$. 4. $(5x - 3y)(6x - 5y)$.

2. $(9r + 7s)(2r - 5s)$. 5. $(2a - 3b)(4a - 5b)$.

3. $(8p - 3q)(5p + 7q)$. 6. $(7x - 5y)(3x + 4y)$.

7. $(6x - 7y)(2x - 3y)$.
8. $(a - 6b)(a - 7b)$.
9. $(6a - 9)(6a + 9)$.
10. $(9u + 2v)(4u - 7v)$.
11. $(9p - 4q)(2p - 3q)$.
12. $(8r - 3s)(2r + 5s)$.
13. $(9x - 4y)(3x - 7y)$.
14. $(6xy + 7z)(-3xy - 4z)$.
15. $(-ax - y)(-ax + y)$.
16. $(abc + 5)(6abc - 7)$.
17. $(9abc - 4a^2b^2c^2)(9abc + 4a^2b^2c^2)$.
18. $(x^2 + x + 1)(x - 1)$.
19. $(-3a + 5a^2 + 2)(-5a - 4)$.
20. $(3x^2 - 4 - 5x)(2 - 3x)$.
21. $(7 - 9r + 3r^2)(5 - 8r)$.
22. $(7r^2 - 8r + 5)(7 - 9r)$.
23. $(5a^2 - 3ab + 2b^2)(3a^2 - 5ab - 4b^2)$.
24. $(4x^2 + 1 - x)(x^2 - 3x + 9)$.
25. $(9x^2 - 7xy + 8y^2)(9x^2 - 8y^2 + 7xy)$.
26. $(6u^2 + 4v^2 - 5uv)(6u^2 + 5uv - 4v^2)$.
27. $(a + a^3 + 1 + a^2)(a - 1)$.
28. $(a^2 - 2ab + b^2)(a^2 + 2ab + b^2)$.
29. $(3a - 4b - 5c)(4a + 5b - 3c)$.
30. $(5x - 7y + 9z)(5x + 7y - 9z)$.
31. $(6r + 5s - 4t)(6r - 5s + 4t)$.
32. $(7a - b - 2c)(-7a + b - 2c)$.
33. $(x^4 - 3x^3 - 4x^2 + 5x + 3)(x - 7)$.
34. $(a - 5a^3 + 3a^2 + 1)(a^2 + 1 + a)$.
35. $(a^3 - 2a^4 - 1)(a - a^5 + 1)$.
36. $(a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4)(a^2 - 2ab + b^2)$.
37. $(-x^3 + x^2y^6 - x^6y^2 + y^8)(x^2y^2 + x^4 + y^4)$.
38. $(x^7 + x^3 + x + 1)(x^3 - 3x - 1)$.
39. $(x^2 - x^5y^3 - x^4 + y)(x^3 - y + x)$.
40. $(x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)(x - 1)$.

41. $(a^2 + \frac{1}{2}a + \frac{1}{3})(2a + \frac{1}{2})$. 47. $(a^{p+q} + 2b)(a^{p+q} - 2b)$.
 42. $(\frac{1}{2}x^2 - \frac{1}{2}x - \frac{1}{4})(\frac{1}{2}x + \frac{1}{4})$. 48. $(a^m - 3a^{m-1} + 4a^{m-2})(a+1)$.
 43. $(a^n + 1)(a^n - 1)$. 49. $(a - 2b)^2$.
 ✓ 44. $(x^n + y^m)(x^n - y^m)$. 50. $(a + b - 2c)^2$.
 ✓ 45. $(x^{n-1} - 1)(x - 1)$. 51. $(a - 2b + c - 3d)^2$.
 ✓ 46. $(x^{2n} + x^ny^m + y^{2m})(x^n - y^m)$. 52. $(a^2 - 1)(a + 1)(a - 1)$.
 53. $(x + y)(x - y)(x - y)(x + y)$.
 54. $(a - b)(a^2 - 2ab + b^2)(a + b)$.
 55. $(x^2 + y^2)(x^2 - y^2)(x^4 - y^4)$.
 56. $(a + 1)(a - 1)(a^2 + 1)(a^4 + 1)(a^8 + 1)$.
 57. $(m - n)(m + n)(m^2 + mn + n^2)(m^2 - mn + n^2)$.
 58. $(p + q)(p^4 + p^3q + p^2q^2 + pq^3 + q^4)(p - q)$.
 59. $(a - b)^3$. 61. $(m - n)^4$.
 60. $(x + 2y)^3$. 62. $(m + 2n + 3p)^3$.
 63. $(\frac{1}{2}a^2 + \frac{1}{4}a - 1)(2a - 4)(6a - 2)$.

If $M = a + b + c$, $N = a + b - c$, $P = a - b + c$, and $Q = -a + b + c$, find the following product:

64. MN . 65. PQ . 66. NPQ .

(For method of detached coefficients see Appendix I.)

SPECIAL CASES IN MULTIPLICATION

63. The product of two binomials which have a common term.

$$(x + 2)(x + 4) = x^2 + 2x + 4x + 8 = x^2 + 6x + 8.$$

$$(x - 2)(x + 4) = x^2 - 2x + 4x - 8 = x^2 + 2x - 8.$$

$$(x - 2)(x - 4) = x^2 - 2x - 4x + 8 = x^2 - 6x + 8.$$

$$\begin{aligned}
 (4a + 7b)(4a - 5b) &= 16a^2 + 28ab - 20ab - 35b^2 \\
 &= 16a^2 + 8ab - 35b^2.
 \end{aligned}$$

64. *The product of two binomials which have a common term is equal to the square of the common term, plus the sum of the two unequal terms multiplied by the common term, plus the product of the two unequal terms.*

$(5a - 6b)(5a - 9b)$ is equal to the square of the common term, $25a^2$, plus the sum of the unequal terms multiplied by the common term, i.e. $(-15b)(5a) = -75ab$, plus the product of the two unequal terms, i.e. $+54b^2$. Hence the product equals $25a^2 - 75ab + 54b^2$.

EXERCISE 23

Multiply by inspection :

- | | |
|--------------------------|--|
| 1. $(x + 3)(x + 5)$. | 16. $(x + y)(x - y)$. |
| 2. $(a - 4)(a - 3)$. | 17. $(a^2 + 5)(a^2 - 6)$. |
| 3. $(a - 6)(a - 4)$. | 18. $(a^2 - 5b)(a^2 - 6b)$. |
| 4. $(a - 6)(a + 4)$. | 19. $(a^2b^2 + 9)(a^2b^2 - 7)$. |
| 5. $(a - 1)(a - 2)$. | 20. $(a^2b^2 - 2c^2)(a^2b^2 + 2c^2)$. |
| 6. $(a - 8)(a - 4)$. | 21. $(a^4 - 7b^4)(a^4 - 7b^4)$. |
| 7. $(x - 12)(x + 13)$. | ✓ 22. $(1 + 2x)(1 + 3x)$. |
| 8. $(x - 12)(x - 13)$. | ✓ 23. $(a^n + 2b^n)(a^n - 2b^n)$. |
| 9. $(x + 3)(x + 3)$. | 24. $(100 + 3)(100 + 1)$. |
| 10. $(x - 25)(x - 4)$. | 25. $(1000 - 6)(1000 + 7)$. |
| 11. $(x + 25)(x - 4)$. | 26. 103×105 . |
| 12. $(x + 2y)(x + 3y)$. | 27. 107×109 . |
| 13. $(x - 4y)(x - 2y)$. | 28. 1008×1009 . |
| 14. $(a - 5b)(a + 4b)$. | 29. 999×1016 . |
| 15. $(a - 8)(a + 8)$. | |

30. $47,796 \times 28,534 = 1,363,811,064$. Find the product of the two next higher numbers. (47,797 and 28,535.)

31. A garden 285 yards long and 215 yards wide is lengthened and widened by one yard. Without multiplying 285 and 215, find the increase in area.

32. $(2a + 7)(2a - 3)$. 40. $(6 + x^2)(x^2 - 7)$.
 33. $(4a + 9)(4a - 8)$. 41. $(x - 1)(x + 1)(x^2 - 3)$.
 34. $(3a - 4)(3a + 6)$. 42. $(\overline{x + y + 7})(\overline{x + y - 3})$.
 35. $(-a + 4)(-a - 4)$. 43. $(x + y - 4)(x + y - 3)$.
 36. $(-x^3 + 4x)(-x^3 - 7x)$. 44. $(a + b)^2$.
 37. $(2 - 7x)(2 - 6x)$. 45. $(a - b)^2$.
 38. $(x - 4)(5 + x)$. 46. $(a + b)(a - b)$.
 39. $(x - 2b)(3b + x)$.
 47. Find two binomials whose product equals $x^2 - 5x + 6$.
 48. Find two factors of $x^2 + 9x + 20$.
 49. Find two factors of $x^2 - 11x + 18$.
-

65. Some special cases of the preceding type of examples deserve special mention:

I. $(a + b)^2 = a^2 + 2ab + b^2$.

II. $(a - b)^2 = a^2 - 2ab + b^2$.

III. $(a + b)(a - b) = a^2 - b^2$.

Expressed in general language:

I. *The square of the sum of two numbers is equal to the square of the first, plus twice the product of the first and the second, plus the square of the second.*

II. *The square of the difference of two numbers is equal to the square of the first, minus twice the product of the first and the second, plus the square of the second.*

III. *The product of the sum and the difference of two numbers is equal to the difference of their squares.*

The student should note that the second type (II) is only a special case of the first (I).

Ex. $(4x^3 + 7y^2)^2$ is equal to the square of the first, i.e. $16x^6$, plus twice the product of the first and the second, i.e. $56x^3y^2$, plus the square of the second, i.e. $49y^4$. Hence the required square equals $16x^6 + 56x^3y^2 + 49y^4$.

EXERCISE 24

Multiply by inspection :

1. $(x + y)^2$.
2. $(a + 4)^2$.
3. $(p - q)^2$.
4. $(n - 4)^2$.
5. $(s - 7)^2$.
6. $(a + 1)^2$.
7. $(m^2 + 2)^2$.
8. $(d^4 - 7)^2$.
9. $(a + 2b)^2$.
10. $(x - 7y)^2$.
11. $(d^2 - 4c^2)^2$.
12. $(m^2 - 12n^2)^2$.
13. $(2x - 7y)^2$.
14. $(2a + 9b)^2$.
15. $(4m - 3n)^2$.
16. $(2x^3 - 3y^2)^2$.
17. $(4a^2b^2 - 7)^2$.
18. $(4d^2e^2 + 3f^2)^2$.
19. $(6x^2y^2 - 3y^2z^2)^2$.
20. $(7mnp^3q - 3n^4p^4q^4)^2$.
21. $(m^n - 1)^2$.
22. $(x^n - y^m)^2$.
23. $(9n + \frac{1}{2})^2$.
24. $(x + y)(x - y)$.
25. $(a^2 + 4)(a^2 - 4)$.
26. $(pq + q)(pq - q)$.
27. $(5s^2 - t^2)(5s^2 + t^2)$.
28. $(6m^3n^3 + 4)(6m^3n^3 - 4)$.
29. $(x^2y^2 - 4z)(x^2y^2 + 4z)$.
30. $(4x^3y^3z^3 - p^4)(4x^3y^3z^3 + p^4)$.
31. $(x^m + y^m)(x^m - y^m)$.
32. $(x^{3m} + 11)(x^{3m} - 11)$.
33. $(x^2 - 7y^2)(x^2 + 7y^2)$.
34. $(m^2 - 2n^4)(2n^4 + m^2)$.
35. $(a^2 + 2ab)(2ab + a^2)$.
36. $(d^4 - 7)(-7 + d^4)$.
37. $(u^7 - 7)(7 + u^7)$.
38. $(cd^2 - 12f)(-12f + cd^2)$.
39. $[(a + b) - c][(a + b) + c]$.
40. $[m + n - p][m + n + p]$.
41. $(a - 2b + 3c)(a - 2b - 3c)$.
42. $(40 + 1)^2$.
43. $(100 - 3)^2$.
44. 21^2 .
45. 102^2 .
46. 51^2 .
47. 99^2 .
48. 203^2 .
49. 997^2 .
50. 53^2 .
51. 54^2 .
52. 96^2 .
53. $(40 + 2)(40 - 2)$.
54. 41×39 .
55. 57×63 .
56. 997×1003 .
57. 6.5×7.5 .
58. Prove that $(n + \frac{1}{2})^2 = n(n + 1) + \frac{1}{4}$.
59. Illustrate Ex. 58 by numerical examples.

66. The product of two binomials whose corresponding terms are similar.

By actual multiplication, we have

$$\begin{array}{r} 3x + 2y \\ 5x - 4y \\ \hline 15x^2 + 10xy \\ \quad - 12xy - 8y^2 \\ \hline 15x^2 - 2xy - 8y^2 \end{array}$$

The middle term of the result is obtained by adding the product of $5x \cdot 2y$ and $-4y \cdot 3x$. These products are frequently called the cross products.

$$\begin{array}{r} 3x + 2y \\ \quad \times \\ 5x - 4y \\ \hline 10xy - 12xy \end{array}$$

Hence in general, *the product of two binomials whose corresponding terms are similar is equal to the product of the first two terms, plus the sum of the cross products, plus the product of the last terms.*

EXERCISE 25

Multiply by inspection:

- | | |
|----------------------------|-----------------------------------|
| 1. $(2x + 3)(x - 5)$. | 7. $(a^2 - 7b^2)(2a^2 - 3b^2)$. |
| 2. $(3x - 2)(x - 7)$. | 8. $(4a^2b^3 - 2)(3a^2b^3 + 5)$. |
| 3. $(3x - 2)(3x - 3)$. | 9. $(2x^2 - 3y^3)(2x^2 - 5y^3)$. |
| 4. $(6x - 1)(2x + 3)$. | 10. $(2x^2y - a)(2x^2y - 7a)$. |
| 5. $(5a - b)(a - 2b)$. | 11. $(-x^3 + 2ab)(2x^3 - 3ab)$. |
| 6. $(7mn + p)(2mn - 3p)$. | 12. $(5x - 7y)(5y - 7x)$. |

67. The square of a polynomial.

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$$

$$\begin{aligned} (a - b + c - d)^2 \\ = a^2 + b^2 + c^2 + d^2 - 2ab + 2ac - 2ad - 2bc + 2bd - 2cd. \end{aligned}$$

The square of a polynomial is equal to the sum of the squares of each term increased by twice the product of each term with each that follows it.

The student should note that the square of each term is always positive, while the product of the terms may have the plus sign or the minus sign.

EXERCISE 26

Find by inspection :

- | | |
|-------------------------|---------------------------------|
| 1. $(x + y + z)^2$. | 7. $(2a^2 - 3b^2 - 3c^2)^2$. |
| 2. $(m + n - p)^2$. | 8. $(x + y + u - 3)^2$. |
| 3. $(m - n - 3)^2$. | 9. $(2a - 2b + 2c - d)^2$. |
| 4. $(m + 2n - 4)^2$. | 10. $(3x - 4y - z + 2u)^2$. |
| 5. $(d^2 - 2e + 1)^2$. | 11. $(x^3 - 2x^2 + 3x - 7)^2$. |
| 6. $(2d - 3e - 4f)^2$. | |

68. In **simplifying a polynomial** the student should remember that a parenthesis is understood about each term. Hence after multiplying the factors of a term, the beginner should inclose the product in a parenthesis.

Ex. Simplify $(x + 6)(x - 4) - (x - 3)(x - 5)$.

Check. If $x = 1$,

$$\begin{aligned}
 (x + 6)(x - 4) - (x - 3)(x - 5) &= (7 \cdot -3) - (-2 \cdot -4) = -29. \\
 &= [x^2 + 2x - 24] - [x^2 - 8x + 15] \\
 &= x^2 + 2x - 24 - x^2 + 8x - 15 \\
 &= 10x - 39. \qquad \qquad \qquad = 10 - 39. \qquad \qquad \qquad = -29.
 \end{aligned}$$

EXERCISE 27

Simplify the following expressions, and check the answers :

1. $(x - 7)(x - 8) - (x - 14)(x - 4)$.
2. $3(x - 9) - 5(x + 7)$.
3. $(a + 7)(5a + 3) - 2(a^2 - 3a + 7)$.
4. $(a + b)^2 - (a - b)^2$.

5. $(m+2)(m-2) + (m-7)(m-2) - 3(m^2-3m)$.
6. $12(a+1) + 17(a-2) - 5(a-3)(a-2)$.
- ✓ 7. $(d+e)(d-e) + (d-e)^2 - (d+e)^2$.
- ✓ 8. $3(x+y)^2 - (3x-y)(3x+y) - 2(2y^2+3xy)$.
9. $(a+b)(a+c) - (b+c)(b+a) + (c+a)(c+b)$.
10. $(a-2b)^2 + (b-2c)^2 + (c-2a)^2$.
11. $(x+y)(x-y) + (y+z)(y-z) + (z+x)(z-x)$.
12. $(a+b+c)^2 - (a+b+c)(a+b-c)$.
13. $a(a+b) - [(a-b)^2 + b(a-b)]$.
14. $(a-b)^2 + (b-c)^2 + (c-a)^2 - (a+b+c)^2$.
15. $4(x-4y)(x+4y) - 2(x-4y)^2 - 2(x^2+8y^2)$.
16. $4(a-b)(a+b) - 4(a-2b)(a-b) + 12b(a-b)$.
17. $4(a+b)(b-a) - 3(b-a)(a+b) - a(a-b)$.
18. $(5x+6y)^2 - (3x-7y)^2 + (4x-5y)(4x+5y)$.
19. $(7x-8y)^2 - (4x-y)^2 - (3x+4y)(3x-4y)$.
20. $(3x-7)^2 - (5x-2)^2 - (6x+7)(7-6x)$.
21. $4(9p-8q)(9p+8q) - 2(7p-3q)^2 - (2p-5q)^2$.
22. $3(x-17)(x-13) - 4(x-9)(x-17) + (x-13)(x-9)$.
- ✓ 23. $(x+5)^2 - (x-9)^2 - 140$.
24. $5x(3x-11) + 2(4x-7)^2 - 75x^2 - 7(9+2x)^2 + 167$.
25. $(3ax+4b)^2 + (4ax-3b)^2 - (5ax-7b)^2 + 11b^2$.
26. $9(5ax-3b)^2 - 4(6ax-7b)^2 - (9ax+5b)(9ax-5b) - 21abx$.
27. $(2.5x+3)(5-2x) + (1.5-x)(4+2x) + 7(x-4)^2$.
28. $(mn+1)^3 - 3mn(mn+1)$.
29. $(x+1)(x+2)(x+3) - (x+2)(x+3)(x+4) - 3(x+2)(x+3)$.
- ✓ 30. $(p+2n)^2 - 3(p-5n)(p+7n)$.

31. $(x^3 + x^2 + x + 1)(x - 1) - (x^2 + 1)(x + 1)(x - 1)$.
32. $a + 3[4 - 3(2 - a)]$.
33. $5d - (6d - 5) + [6 - 4(d - 9)]$.
34. $12mn - 5(2mn - 7) - [3mn - 4(mn - 2) - (mn - 2)]$.
35. $\{(a + x)^2 - (a - x)^2\} - 2\{(a + x)(a - x) - (a - x)^2\}$.
36. $(2x - 1)(3x + 5)(x + 1) - (x - 1)(x - 2)(x + 3)$.
37. $(3x + 5)(2x - 3)(x - 1) - (x - 1)(x + 2)(x - 3)$.
38. $3(a + 2b)(a - b) - 4(2a - 3b)(3a + b) + b(2a - 9b)$.
39. $x - (9a + 8b - 7c)(f + g) + (10a + 8b - 7c)(f - g)$.
40. $(3a - 6c)(4a - 3d) - \{(2a - 5c)(6a - 11d) - (37cd - 6ac)\}$.

✓ 41. From 0 subtract three times the difference between the squares of $(a + b)$ and $(a - b)$ and add the result to $6ab$.

CHAPTER IV

DIVISION

69. Division is the process of finding one of two factors if their product and the other factor are given.

The **dividend** is the product of the two factors, the **divisor** is the given factor, and the **quotient** is the required factor.

Thus to divide -12 by $+3$, we must find the number which multiplied by $+3$ gives -12 . But this number is -4 ; hence $\frac{-12}{+3} = -4$.

70. Since

$$+a \cdot +b = +ab$$

$$+a \cdot -b = -ab$$

$$-a \cdot +b = -ab$$

and

$$-a \cdot -b = +ab,$$

it follows that

$$\frac{+ab}{+a} = +b$$

$$\frac{-ab}{+a} = -b$$

$$\frac{-ab}{-a} = +b$$

$$\frac{+ab}{-a} = -b.$$

71. Hence the **law of signs** is the same in division as in multiplication: *Like signs produce plus, unlike signs minus.*

72. Law of Exponents. It follows from the definition that $a^8 \div a^5 = a^3$, for $a^3 \times a^5 = a^8$.

Or in general, if m and n are positive integers, and m is greater than n , $a^m \div a^n = a^{m-n}$, for $a^{m-n} a^n = a^m$.

The exponent of a quotient of two powers with equal bases equals the exponent of the dividend diminished by the exponent of the divisor.

DIVISION OF MONOMIALS

73. To divide $10 x^7 y^3 z$ by $-2 x^5 y^2$, we have to find the number which multiplied by $-2 x^5 y^2$ gives $10 x^7 y^3 z$. This number is evidently $-5 x^2 yz$.

Therefore,
$$\frac{10 x^7 y^3 z}{-2 x^5 y^2} = -5 x^2 yz.$$

Hence, the quotient of two monomials is a monomial whose coefficient is the quotient of their coefficients, preceded by the proper sign, and whose literal part is the quotient of their literal parts found in accordance with the law of exponents.

EXERCISE 28

Perform the divisions indicated:

1. $-64 \div -4.$
2. $64 \div -4.$
3. $\frac{44 a}{-11 a}.$
4. $\frac{-17 ab^2}{17 b^2}.$
5. $\frac{6 a^3 b^3 c^3}{-2 ab^2 c^3}.$
6. $\frac{-75 ab^3 c^5}{15 ac^3}.$
7. $\frac{-12 x^5 y^5 z^5}{-3 x^5 yz^2}.$
8. $\frac{98 p^7 q r^7}{-7 q}.$
9. $\frac{-20 a^2 x^{21} y^2}{-20 a^2 x^{20}}.$
10. $\frac{154 m^{14} n^{21} p}{-11 mp}.$
11. $\frac{(a+b)^5}{(a+b)^3}.$
12. $\frac{84 x^2 y^7 z^6}{-7 xy^2 z}.$
13. $\frac{28 m^7 n^8 p^7}{4 m^6 n^7 p^4}.$
14. $\frac{-16 mp^2 q^7}{-16 mp^2 q^7}.$
15. $\frac{96 m^{20} n^{21} x^{40}}{12 m^{20} n^{19} x}.$
16. $\frac{-144 a^7 c^3 d}{-16 a^7 c}.$
17. $\frac{50 d^4 e^4 f}{-50 d^4 e^4}.$
18. $\frac{(a-b)^3}{a-b}.$
19. $\frac{6(a+b-c)^7}{-2(a+b-c)^6}.$
20. $\frac{40 a^3 m y^n}{-20 a^m y^n}.$

DIVISION OF POLYNOMIALS BY MONOMIALS

74. To divide $ax + bx + cx$ by x we must find an expression which multiplied by x gives the product $ax + bx + cx$.

But
$$x(a + b + c) = ax + bx + cx.$$

Hence
$$\frac{ax + bx + cx}{x} = a + b + c.$$

To divide a polynomial by a monomial, divide each term of the dividend by the monomial and add the partial quotients thus formed.

E.g.
$$\frac{-6x^3yz^4 - 15xy^2z^3 + 3xyz^2}{-3xyz^2} = 2x^2yz^2 + 5yz - 1.$$

EXERCISE 29

Perform the operations indicated:

1. $\frac{56ax - 63bx}{7x}.$

4. $\frac{4x^4 - 3x^2 + 5x}{-x}.$

2. $\frac{-26a^3c^2 + 39a^3d^2}{13a^3}.$

5. $\frac{-9a^6 + 6a^3 - 3a^9}{-3a^3}.$

3. $\frac{-14a^2b^2c^3 + 21a^4b^2c^2}{-7a^2b^2c^2}.$

6. $\frac{-11x^3y + 22x^2y^2 - 33xy^3}{11xy}.$

7. $\frac{4a^2x^2 - 40a^3x^3 + 8a^4x^4}{4a^2}.$

8. $\frac{-12a^2b^3c^3 + 6a^2b^3c^4 - 18a^3b^3c^5}{6a^2b^3c^3}.$

9. $\frac{25p^6 - 20p^4 - 5p^2}{-5p^2}.$

10. $\frac{49m^2n^2x - 28m^3n^3y + 7m^4n^4}{-7m^2n^2}.$

11. $\frac{169x^3y^3z^3 - 26x^2y^2z^3 + 39x^2y^3z^2}{13x^2y^2z^2}.$

12. $(92r^5s^5 - 115r^3s^4 - 161r^2s^6 + 69r^6s^3) \div 23rs.$

13. $(187x^4y^2 - 121x^3y^3 - 88x^2y^4 + 231x^2y^5) \div 11xy.$

$$14. (75ab^2 - 105a^2b - 165a^2b^2 + 180ab) \div 15ab.$$

$$15. (6.8ab - 8.5ac - 5.1ad + 3.4ae) \div 1.7a.$$

$$16. \frac{6(a+b)^3 + 15(a+b)^2}{-3(a+b)^2}.$$

$$18. \frac{a^{3n} + a^{2n} + a^n}{a^n}.$$

$$17. \frac{4(x-y) - 8(x-y)^3}{-2(x-y)}.$$

$$19. \frac{x^{3m}y^n + x^{2m}y^{2n} + x^my^{3n}}{-x^ny^n}.$$

$$20. \frac{m^{a+4} + m^{a+3} + m^{a+2}}{m^{a+1}}.$$

DIVISION OF A POLYNOMIAL BY A POLYNOMIAL

75. Let it be required to divide $25a - 12 + 6a^3 - 20a^2$ by $2a^2 + 3 - 4a$, or, arranging according to descending powers of a , divide $6a^3 - 20a^2 + 25a - 12$ by $2a^2 - 4a + 3$.

The term containing the highest power of a in the dividend (*i.e.* $6a^3$) is evidently the product of the terms containing respectively the highest power of a in the divisor and in the quotient.

Hence the term containing the highest power of a in the quotient is $\frac{6a^3}{2a^2}$, or $3a$.

If the product of $3a$ and $2a^2 - 4a + 3$, *i.e.* $6a^3 - 12a^2 + 9a$, be subtracted from the divisor, the remainder is $-8a^2 + 16a - 12$.

This remainder obviously must be the product of the divisor and the rest of the quotient. To obtain the other terms of the quotient we have therefore to divide the remainder, $-8a^2 + 16a - 12$, by $2a^2 - 4a + 3$.

We consequently repeat the process. By dividing the highest term in the new dividend $-8a^2$ by the highest term in the divisor $2a^2$, we obtain -4 , the next highest term in the quotient.

Multiplying -4 by the divisor $2a^2 - 4a + 3$ we obtain the product $-8a^2 + 16a - 12$, which subtracted from the preceding dividend leaves no remainder.

Hence $3a - 4$ is the required quotient.

The work is usually arranged as follows :

$$\begin{array}{r|l}
 6a^3 - 20a^2 + 25a - 12 & 2a^2 - 4a + 3 \\
 \underline{6a^3 - 12a^2 + 9a} & \underline{3a - 4} \\
 -8a^2 + 16a - 12 & \\
 \underline{-8a^2 + 16a - 12} &
 \end{array}$$

76. The method which was applied in the preceding example may be stated as follows:

1. *Arrange dividend and divisor according to ascending or descending powers of a common letter.*

2. *Divide the first term of the dividend by the first term of the divisor, and write the result for the first term of the quotient.*

3. *Multiply this term of the quotient by the whole divisor, and subtract the result from the dividend.*

4. *Arrange the remainder in the same order as the given expression, consider it as a new dividend and proceed as before.*

5. *Continue the process until a remainder zero is obtained, or until the highest power of the letter according to which the dividend was arranged is less than the highest power of the same letter in the divisor.*

77. Checks. Numerical substitution constitutes a very convenient, but not absolutely reliable check.

An absolute check consists in multiplying quotient and divisor. The result must equal the dividend if the division was exact, or the dividend diminished by the remainder if the division was not exact.

Ex. 1. Divide $8a^3 + 8a - 4 + 6a^4 - 11a^2$ by $3a - 2$.

Arranging according to descending powers,

$ \begin{array}{r} 6a^4 + 8a^3 - 11a^2 + 8a - 4 \\ \underline{6a^4 - 4a^3} \\ + 12a^3 - 11a^2 \\ + 12a^3 - 8a^2 \\ \hline - 3a^2 + 8a \\ - 3a^2 + 2a \\ \hline + 6a - 4 \\ + 6a - 4 \\ \hline 0 \end{array} $	<div style="text-align: right;"> <i>Check.</i> If $a = b = 1$, $= 7 \div 1$ $= 7.$ </div> <div style="border-left: 1px solid black; padding-left: 10px; margin-left: 10px;"> $\begin{array}{r} 3a - 2 \\ \hline 2a^3 + 4a^2 - a + 2 \end{array}$ </div>
--	---

Ex. 2. Divide $a^4 - 4b^4 - 6a^3b + 9a^2b^2$ by $2b^2 - 3ab + a^2$.

Arranging according to descending powers of a , we have

$$\begin{array}{r}
 a^4 - 6a^3b + 9a^2b^2 - 4b^4 \\
 \underline{a^4 - 3a^3b + 2a^2b^2} \\
 -3a^3b + 7a^2b^2 - 4b^4 \\
 \underline{-3a^3b + 9a^2b^2 - 6ab^3} \\
 -2a^2b^2 + 6ab^3 - 4b^4 \\
 \underline{-2a^2b^2 + 6ab^3 - 4b^4}
 \end{array}
 \quad
 \begin{array}{r}
 a^2 - 3ab + 2b^2 \\
 \hline
 a^2 - 3ab - 2b^2
 \end{array}$$

Check. The numerical substitution $a = 1, b = 1$, cannot be used in this example since it renders the divisor zero. Hence we have either to use a larger number for a , or multiply.

$$\begin{aligned}
 & (a^2 - 3ab + 2b^2)(a^2 - 3ab - 2b^2) \\
 &= [(a^2 - 3ab) + 2b^2][(a^2 - 3ab) - 2b^2] \\
 &= (a^2 - 3ab)^2 - 4b^4 \\
 &= a^4 - 6a^3b + 9a^2b^2 - 4b^4.
 \end{aligned}$$

Ex. 3. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.

Check. If $a = b = 1$,

$$\begin{array}{r}
 a^3 - 3abc + b^3 + c^3 \\
 \underline{a^3 + a^2b + a^2c} \\
 -a^2b - a^2c - 3abc + b^3 + c^3 \\
 \underline{-a^2b - ab^2 - abc} \\
 -a^2c + ab^2 - 2abc + b^3 + c^3 \\
 \underline{-a^2c \quad -abc - ac^2} \\
 ab^2 - abc + ac^2 + b^3 + c^3 \\
 \underline{ab^2 \quad \quad \quad + b^3 + b^2c} \\
 -abc + ac^2 - b^2c + c^3 \\
 \underline{-abc \quad -b^2c - bc^2} \\
 ac^2 + bc^2 + c^3 \\
 \underline{ac^2 + bc^2 + c^3}
 \end{array}
 \quad
 \begin{array}{r}
 a + b + c \\
 \hline
 a^2 - ab - ac + b^2 - bc + c^2 \\
 \hline
 \end{array}
 \begin{array}{l}
 = 0 \div 3 \\
 = 0.
 \end{array}$$

Ex. 4. Divide $a^{m+3} - 2a^{m+1} - a^{m-1}$ by $a^{m+1} + 2a^m + a^{m-1}$.

$$\begin{array}{r}
 a^{m+3} - 2a^{m+1} + a^{m-1} \\
 \underline{a^{m+3} + 2a^{m+2} + a^{m+1}} \\
 -2a^{m+2} - 3a^{m+1} + a^{m-1} \\
 \underline{-2a^{m+2} - 4a^{m+1} - 2a^m} \\
 a^{m+1} + 2a^m + a^{m-1} \\
 \underline{a^{m+1} + 2a^m + a^{m-1}}
 \end{array}
 \quad
 \begin{array}{r}
 a^{m+1} + 2a^m + a^{m-1} \\
 \hline
 a^2 - 2a + 1
 \end{array}$$

EXERCISE 30

Divide:

1. $x^2 - 7x + 12$ by $x - 4$.
2. $a^2 - a - 72$ by $a - 9$.
3. $6a^2 + 23ab + 20b^2$ by $2a + 5b$.
4. $15b^2 - 19bc - 56c^2$ by $3b - 8c$.
5. $42x^2 - 10xy - 12y^2$ by $7x + 3y$.
6. $4a^2 - 121b^2$ by $2a - 11b$.
7. $21a^2 + 13ab - 20b^2$ by $7a - 5b$.
8. $16r^2 - 46rs + 15s^2$ by $2r - 5s$.
9. $40u^2 - 53uv + 6v^2$ by $5u - 6v$.
10. $18p^2 + 19pq - 12q^2$ by $2p + 3q$.
11. $90m^2 - 281mn - 85n^2$ by $5m - 17n$.
12. $72p^2 + 7pq - 2q^2$ by $8p - q$.
13. $x^3 + y^3$ by $x + y$.
14. $x^3 - y^3$ by $x - y$.
15. $2a^3 - 9a^2 + 11a - 3$ by $2a - 3$.
16. $13p + 1 + 47p^2 + 35p^3$ by $5p + 1$.
17. $4a^3 - 24a - 9 - 3a^2$ by $a - 3$.
18. $2a^3 + 18 - 3a - 7a^2$ by $2a + 3$.
19. $x^4 - 16y^4$ by $x - 2y$.
20. $9a^2 + 24ab + 16b^2 - 36c^2$ by $3a + 4b + 6c$.
21. $16x^2 - 40xy + 25y^2 - 9z^2$ by $4x - 5y - 3z$.
22. $16 + 40a + 25a^2 - 49a^4$ by $4 + 5a - 7a^2$.

Perform the operations indicated and check the answers:

23. $(216a^3 + 125) \div (36a^2 - 30a + 25)$.
24. $(128a^4b^3 - 160a^5b^2 + 2a^6b + 15a^7) \div (3a^2 - 8ab)$.
25. $(1 - 32p^5) \div (1 + 2p + 4p^2 + 8p^3 + 16p^4)$.

26. $(81x^2y^2 + 72y^2 - 63xy^2 + 56y - 49xy + 63x^2y) \div (9x^2 + 8 - 7x)$.
27. $(1 + 81a^4 - 18a^2) \div (1 - 6a + 9a^2)$.
28. $(15x^4 + 7x + 7x^3 + 15x^2 + 4) \div (3x^2 + 2x + 1)$.
29. $(7x^3 - 2x^4 + 82x^2 + 145x + 72) \div (9 + 8x - x^2)$.
30. $(15a^4 - 34a^3 + a^2 + 2a - 8) \div (3a^2 - 5a - 4)$.
31. $(42m^4 + 174m^3 + 70 - 338m + 4m^2) \div (3m^2 + 3m - 7)$.
32. $(18p^4 - 9pq^3 - 33p^3q + 32p^2q^2 - 8q^4) \div (6p^2 - 7pq + 8q^2)$.
33. $(56a^4 + 119a^2b^2 + 45b^4 - 53a^3b - 47ab^3) \div (7a^2 + 9b^2 - 4ab)$.
34. $(x^5 - y^5) \div (x - y)$. 36. $(a^4 - b^4) \div (a + b)$.
35. $(8x^6 - 27y^3) \div (2x^2 - 3y)$. 37. $(a^5 + b^5) \div (a + b)$.
38. $(81a^4b^4 - 625c^4) \div (3ab - 5c)$.
39. $(a^4 + a^2b^2 + b^4) \div (a^2 - ab + b^2)$.
40. $(4a^4 - 13a^2b^2 + 9b^4) \div (2a^2 + ab - 3b^2)$.
41. $(a^3 + 3a^2b + 3ab^2 + b^3 + c^3) \div (a + b + c)$.
42. $(a^3 - 3ab + b^3 + 1) \div (a + b + 1)$.
43. $(a^2 - 6ab + 9b^2 - c^2 - 2cd - d^2) \div (a - 3b - c - d)$.
44. $(a^3 - 3a^2b + 3ab^2 - b^3 - 1) \div (a - b - 1)$.
45. $(a^3 + b^3 - 3ab + 1) \div (a^2 + b^2 + 1 - ab - a - b)$.
46. $\frac{x^{3m} - y^{3m}}{x^m - y^m}$. 49. $\frac{6a^m + 23a^{m-1}b + 20a^{m-2}b^2}{2a^{m-1} + 5a^{m-2}b}$.
47. $\frac{x^{4m} + x^{2m}y^{2n} + y^{4n}}{x^{2m} + x^my^n + y^{2n}}$. 50. $\frac{1.4x^2 - 63.08xy + 3.6y^2}{7x - .4y}$.
48. $\frac{a^{m+3} + 3a^{m+2} + 3a^{m+1} + a^m}{a^{m+1} + 2a^m + a^{m-1}}$. 51. $\frac{.3p^2 + .03pq - .06q^2}{.5p - .2q}$.
52. $(p^2 - \frac{1}{12}pq - \frac{1}{2}q^2) \div (\frac{3}{2}p + q)$.

Find four terms of the quotient:

53. $\frac{1}{1+x}$.

54. $\frac{1}{1-x}$.

DIVISION

SPECIAL CASES IN DIVISION

78. Division of the difference of two squares.

Since $(a + b)(a - b) = a^2 - b^2$,

$$\frac{a^2 - b^2}{a - b} = a + b, \text{ and } \frac{a^2 - b^2}{a + b} = a - b.$$

I.e. the difference of the squares of two numbers is divisible by the difference or by the sum of the two numbers.

Ex. 1. $\frac{9x^2 - 16y^2}{3x - 4y} = 3x + 4y.$

Ex. 2. $\frac{(a + b)^2 - (x - y)^2}{(a + b) + (x - y)} = (a + b) - (x - y) = a + b - x + y.$

EXERCISE 31

Write by inspection the quotient of:

1. $\frac{a^2 - 1}{a - 1}.$

2. $\frac{b^2 - 4}{b + 2}.$

3. $\frac{c^2 - 4d^2}{c - 2d}.$

4. $\frac{4x^2 - 9y^2}{2x + 3y}.$

5. $\frac{49x^4y^4 - 64z^2}{7x^2y^2 - 8z}.$

6. $\frac{121x^2y^2z^2 - 4a^2b^2}{11xyz + 2ab}.$

7. $\frac{25a^8 - 1}{5a^4 + 1}.$

8. $\frac{(a + b)^2 - 4x^4}{(a + b) - 2x^2}.$

Find exact binomial divisors of each of the following expressions:

9. $a^4 - b^4.$

10. $x^6 - y^6.$

11. $x^8 - y^4.$

12. $a^{12} - b^{12}.$

13. $x^4y^6 - 625.$

14. $289a^6b^8 - 196.$

15. $169a^4 - b^{100}.$

16. $a^{2m} - b^6.$

17. $(a + b)^2 - 225p^4.$

18. $(a - b)^{4n} - 25q^{10n}.$

19. $(a + b + c)^2 - 1.$

ELEMENTARY ALGEBRA

and two factors of:

20. $10,000 - 81$.

22. $999,991$.

21. 8099 .

23. 63.91 .

79. Division of the sum or the difference of two cubes.

By actual division, we obtain

$$\frac{a^3 - b^3}{a - b} = a^2 + ab + b^2$$

and

$$\frac{a^3 + b^3}{a + b} = a^2 - ab + b^2.$$

The **difference of the cubes** of two numbers divided by the difference of the two numbers equals the square of the first, plus the product of the first and the second, plus the square of the second.

The **sum of the cubes** of two numbers divided by the sum of the two numbers equals the square of the first, minus the product of the first and the second, plus the square of the second.

EXERCISE 32

Write by inspection the quotient of:

1. $(x^3 - y^3) \div (x - y)$.

5. $(m^3 - 125) \div (m - 5)$.

2. $(x^3 + y^3) \div (x + y)$.

6. $(27a^3 + b^3) \div (3a + b)$.

3. $(c^3 + 1) \div (c + 1)$.

7. $(64a^3 + 27b^3) \div (4a + 3b)$.

4. $(m^3n^3 - 1) \div (mn - 1)$.

8. $(a^6 - b^6) \div (a^2 - b^2)$.

EXERCISE 33

Determine what binomial or binomials, if any, will divide each of the following expressions, and find the quotient or quotients:

1. $m^3 + 8$.

4. $x^3y^3 + 27$.

7. $343 - x^6y^6$.

2. $m^3 - 8$.

5. $x^3 - 64$.

8. $512 + x^9$.

3. $x^3y^3 - 27$.

6. $a^3b^3 + c^6$.

9. $x^2 + y^2$.

- | | | |
|-------------------------|----------------------|--------------------------|
| 10. $729x^3 - y^{15}$. | 14. $x^6 + 121$. | 18. $1000x^6y^6 + z^3$. |
| 11. $x^6 + 125$. | 15. $x^2 - y^4$. | 19. $216a^3 - 8b^3$. |
| 12. $x^6 - 125$. | 16. $x^2 - y^3$. | 20. $(x + y)^3 - 8z^3$. |
| 13. $x^6 - 121$. | 17. $a^6 - b^6c^6$. | 21. $a^6 + b^6$. |

Find numerical divisors of the following :

- | | | |
|-------------------|-------------------|--------------------|
| 22. $1000 - 64$. | 24. 1001 . | 26. 8027 . |
| 23. $1000 + 27$. | 25. $1,000,001$. | 27. $64,000,001$. |

CHAPTER V

LINEAR EQUATIONS AND PROBLEMS

80. The **first member** or **left side** of an equation is that part of the equation which precedes the sign of equality. The **second member** or **right side** is that part which follows the sign of equality.

Thus, in the equation $2x + 4 = x - 9$, the first member is $2x + 4$, the second member is $x - 9$.

81. An **identical equation** or **identity** is an equation which is true for all values of the letters involved.

Thus, $(a + b)(a - b) = a^2 - b^2$, no matter what values we assign to a and b . The sign of identity sometimes used is \equiv , thus we may write $(a + b)(a - b) \equiv a^2 - b^2$.

82. An **equation of condition** is an equation which is true only for certain values of the letters involved. An equation of condition is usually called an **equation**.

$x + 9 = 20$ is true only when $x = 11$; hence it is an equation of condition.

83. A set of numbers which when substituted for the letters in an equation produce equal values of the two members, is said to **satisfy** an equation.

Thus $x = 12$ satisfies the equation $x + 1 = 13$. $x = 20$, $y = 7$ satisfy the equation $x - y = 13$.

84. An **equation is employed** to discover an unknown number, usually denoted by x , y , or z , from its relation to known numbers.

85. If an equation contains only one unknown quantity, any value of the unknown quantity which satisfies the equation is a **root** of the equation.

9 is a root of the equation $2x + 2 = 20$.

86. To **solve** an equation is to find its roots.

87. A **numerical equation** is one in which all the known quantities are expressed in arithmetical numbers; as $(7 - x)(x + 4) = x^2 - 2$.

88. A **literal equation** is one in which at least one of the known quantities is expressed by a letter or a combination of letters; as $x + a = bx - c$.

89. A **linear equation** or an equation of the **first degree** is one which when reduced to its simplest form contains only the first power of the unknown quantity; as $9x - 2 = 6x + 7$. A linear equation is also called a *simple equation*.

90. The process of solving equations depends upon the following principles, called **axioms**:

1. *If equals be added to equals, the sums are equal.*
2. *If equals be subtracted from equals, the remainders are equal.*
3. *If equals be multiplied by equals, the products are equal.*
4. *If equals be divided by equals, the quotients are equal.*
5. *Like powers or like roots of equals are equal.* \pm

NOTE. Axiom 4 is not true if the divisor equals zero. *E.g.* $0 \times 4 = 0 \times 5$, but 4 does not equal 5.

91. **Transposition of terms.** A term may be transposed from one member to another by changing its sign.

Consider the equation $x + a = b$.

Subtracting a from both members, $x = b - a$. (Axiom 2)

I.e. the term a has been transposed from the left to the right member by changing its sign.

Similarly, if $x - a = b$.

Adding a to both members, $x = b + a$. (Axiom 1)

The result is the same as if we had transposed a from the first member to the right member and changed its sign.

It follows from § 91 that:

92. *Any term that occurs with the same sign in both members of an equation may be canceled.*

93. *The sign of every term of an equation may be changed without destroying the equality.*

Consider the equation $-x + a = -b + c.$

Multiplying each member by -1 , $x - a = b - c.$ (Axiom 3)

SOLUTION OF LINEAR EQUATIONS

✓ **94. Ex. 1.** Solve the equation $6x - 5 = 4x + 1.$

Transposing $4x$ to the first, and 5 to the second member,

$$6x - 4x = 1 + 5.$$

Uniting similar terms, $2x = 6.$

Dividing both members by 2 , $x = 3.$ (Axiom 4)

Check. When $x = 3.$

The first member, $6x - 5 = 18 - 5 = 13.$

The second member, $4x + 1 = 12 + 1 = 13.$

Hence the answer, $x = 3$ is correct.

Ex. 2. Solve the equation $4 - 3(x - 2) = 6(x + 3) - 6x.$

Simplifying both members, $4 - 3x + 6 = 6x + 18 - 6x.$

Transposing, $-3x - 6x + 6x = -4 - 6 + 18.$

Uniting terms, $-3x = 8.$

Dividing by -3 , $x = -\frac{8}{3}.$

Check. If $x = -\frac{8}{3}.$

The first member, $4 - 3(x - 2) = 4 - 3(-\frac{14}{3}) = 4 + 14 = 18.$

The second member, $6(x + 3) - 6x = 6(\frac{1}{3}) - 6(-\frac{8}{3}) = 2 + 16 = 18.$

95. To solve a simple equation, transpose the unknown terms to the first member, and the known terms to the second. Unite similar terms, and divide both members by the coefficient of the unknown quantity.

Ex. 3. Solve the equation $(4 - x)(5 - x) = 2(11 - 3x) + x^2$.

Simplifying, $20 - 9x + x^2 = 22 - 6x + x^2$.

Canceling x^2 and transposing, $-9x + 6x = -20 + 22$.

Uniting, $-3x = 2$.

Dividing by -3 , $x = -\frac{2}{3}$.

Check. If $x = -\frac{2}{3}$.

The first member, $(4 - x)(5 - x) = (4 + \frac{2}{3})(5 + \frac{2}{3}) = \frac{14}{3} \cdot \frac{17}{3} = \frac{238}{9} = 26\frac{4}{9}$.

The second member, $2(11 - 3x) + x^2 = 2(11 + 2) + \frac{4}{9} = 26 + \frac{4}{9} = 26\frac{4}{9}$.

Ex. 4. Solve the equation $1.3x - [.3 - .5(x - 3)] = 3.8 - x$.

Simplifying, $1.3x - [.3 - .5x + 1.5] = 3.8 - x$,

or, $1.3x - .3 + .5x - 1.5 = 3.8 - x$.

Transposing, $1.3x + .5x + x = .3 + 1.5 + 3.8$.

Uniting, $2.8x = 5.6$.

Check. If $x = 2$, $x = 2$.

The first member, $1.3x - [.3 - .5(x - 3)] = 2.6 - [.3 - .5(-1)]$
 $= 2.6 - .8 = 1.8$.

The second member, $3.8 - x = 3.8 - 2 = 1.8$.

NOTE. The decimals in Ex. 4 could be removed by multiplying each member by 10.

Ex. 5. Solve the equation $\frac{1}{2}(x - 4) = \frac{1}{3}(x + 3)$.

Simplifying, $\frac{1}{2}x - 2 = \frac{1}{3}x + 1$.

Transposing, $\frac{1}{2}x - \frac{1}{3}x = 2 + 1$.

Uniting, $\frac{1}{6}x = 3$.

Dividing by $\frac{1}{6}$, $x = 18$.

Check. If $x = 18$.

The left member $\frac{1}{2}(x - 4) = \frac{1}{2} \times 14 = 7$.

The right member $\frac{1}{3}(x + 3) = \frac{1}{3} \times 21 = 7$.

NOTE. Instead of dividing by $\frac{1}{6}$ both members of the equation $\frac{1}{6}x = 3$, it would be simpler to multiply both members by 6.

EXERCISE 34

Solve the following equations and check the answers:

1. $3x = 17 + 2x.$

8. $14x - 12x = 10 - 3x.$

2. $5x = 3x + 12.$

9. $69 - 7x = 99 - 13x.$

3. $2x = 10 - 3x.$

10. $5x + 36 - 9x = 12.$

4. $7x + 16 = 30.$

11. $9x - 23 = 13 - 3x.$

5. $25 + 3x = 46.$

12. $11x - 72 + 9x = 88 + 10x$

6. $4x - 29 = 11.$

13. $4x - 27 = 1 - 3x.$

7. $16x - 7 = 15x - 3.$

14. $24 - 3x = 25 - 4x.$

15. $25 + 6x - 8x = 17 - 4x + 12.$

16. $20 - 7x + 9 = 40 - 6x + 30.$

17. $7x = 59 - (12x + 21).$

18. $6x - (11x - 10) = 87 - (21 + 13x).$

19. $10x - (4x + 2) = 6 - (21x - 19).$

20. $7(8x - 5) = 77.$

21. $5(2x + 7) = 9(2x - 5).$

22. $6(3x - 8) = 7(5x - 19).$

23. $5x + 7(2x + 3) = 59.$

24. $13x - 2(4x - 5) = 66 - 3x.$

25. $6(5x - 2) - 5(6x - 5) = 4(9 - 2x) + 1.$

26. $7(6x + 5) - 8(3 - 4x) + 24 = 12(2x + 3) + 199.$

27. $x^2 + 13x - (x^2 - x) = 5(2x + 3) + 5.$

28. $x(x + 10) - 6(x - 5) = 4 - (x - x^2).$

29. $(x - 1)(x + 6) = (x + 5)(x - 2).$

30. $(x - 5)(x - 7) = (x + 4)(x - 9) - 13.$

31. $(x + 3)(x - 6) = (x - 5)(x + 6) - 20.$

32. $(x + 9)(x + 2) = (x - 3)(x + 1) + 60.$

33. $(x - 13)(x - 5) = (x - 10)(x - 11).$

34. $(2x + 2)(4x - 3) = (4x - 4)(2x + 3).$
35. $(3x - 5)(2x + 5) - (x + 1)(6x - 4) = 0.$
36. $(4x + 1)(7x + 4) - (14x + 1)(2x - 1) = 285.$
37. $(x + 5)^2 - (x - 9)^2 = 140.$
38. $(x - 7)^2 + (x + 4)^2 = (x - 3)^2 + (x + 5)^2 + 11.$
39. $(x + 4)^2 - (x - 6)^2 = (x + 2)^2 - (x + 3)^2 + 161.$
40. $2(x + 1)^2 - (2x - 3)(x + 2) = 12.$
41. $4(x - 1)(x - 2) - 2(x - 7)(2x + 1) - 6 = 0.$
42. $(5x - 2)(x - 3) - 5(x - 1)(x - 4) + 14 = 0.$
43. $(4y - 7)(9y - 48) - 12(3y + 1)(y - 6) = 0.$
44. $x(x - 1)(x + 7) = (x + 1)(x + 2)(x + 3).$
45. $22x - [3 - (3x - 2)] = 2x - (4x - 3).$
46. $\frac{1}{4}x - 2 = \frac{1}{5}x + 3.$
47. $\frac{1}{2}x + \frac{1}{3}x = 15.$
48. $\frac{1}{4}x + 15 = \frac{1}{5}x - 17.$
49. $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x + \frac{1}{5}x = 1.$
50. $\frac{1}{2}(x - 4) = \frac{1}{3}(x + 2).$
51. $19 - (17 + \frac{1}{8}x) = \frac{1}{2}x + 7.$
52. $4(\frac{2}{5}x - 3) - \frac{3}{10}x = \frac{5}{6}(2x - 16) - 6.$
53. $8.7x + 6.8 = 5x + 7.6 + 3.8x.$
54. $.38x - .18x - (.93 - .29x) = .21 - (.41x - .28).$
55. $x^2 - (.2x + 3)(3x + .4) = (4x + .7)(.1x - 3) - 1.95.$

SYMBOLICAL EXPRESSIONS

96. Suppose one part of 70 to be x , and let it be required to find the other part. If the student finds it difficult to answer this question, he should first attack a similar problem stated

in arithmetical numbers only, *e.g.*: One part of 70 is 25; find the other part. Evidently 45, or $70 - 25$, is the other part. Hence if one part is x , the other part is $70 - x$.

Whenever the student is unable to express a statement in algebraic symbols, he should formulate a similar question stated in arithmetical numbers only, and apply the method thus found to the algebraic problem.

Ex. 1. What must be added to a to produce a sum b ?

Consider the arithmetical question: What must be added to 7 to produce the sum 12?

The answer is 5, or $12 - 7$.

Hence $b - a$ must be added to a to give b .

Ex. 2. $x + y$ yards cost \$100; find the cost of one yard.

If 7 yards cost one hundred dollars, one yard will cost $\frac{\$100}{7}$.

Hence if $x + y$ yards cost \$100, one yard will cost $\frac{100}{x + y}$ dollars.

EXERCISE 35

1. By how much does 7 exceed a ?
2. By how much is $5b$ greater than $3b$?
3. What number exceeds a by 7?
4. What is the 4th part of x ?
5. What is the n th part of a ?
6. By how much does the third part of a exceed the fourth part of b ?
7. By how much does the double of a exceed the tenth part of b ?
8. Two numbers differ by 9, and the smaller one is n . Find the greater one.
9. Divide 20 into two parts such that one part equals x .
10. Divide a into two parts such that one part is 7.

11. What is the dividend if the divisor is a and the quotient is b ?

12. What is the quotient if the dividend is $x + y$ and the divisor is m ?

13. What number divided by 10 will give x as a quotient?

14. By what number must $x + y$ be divided to give 10 as a quotient?

15. What is the dividend if the divisor is d , the quotient is q , and the remainder is r ?

16. Divide a into two parts such that one part is b .

17. How much does a lack of being b ?

18. What is the excess of $a + 12$ over $b + 12$?

19. What is the excess of $a + b$ over $b + c$?

20. The difference between two numbers is d and the smaller one is x . Find the greater one.

21. What number must be subtracted from $2x + 4$, to give a remainder $3x - 5$?

22. A is x years old and B is y years old. How many years is A older than B?

23. A is x years old. How old was he 7 years ago? How old will he be y years hence?

24. If A's age is x years, and B's age is y years, find the sum of their ages 6 years hence.

25. x exceeds an unknown number by y . Find that number.

26. Two numbers differ by $x + y$, and the greater one is $3y$. Find the smaller one.

27. A product consisting of two factors equals a . Find one of the factors, if the other equals $2x$.

28. The smallest of 3 consecutive numbers is x . What are the other two?

29. The greatest of four consecutive numbers is y . Find the other numbers.

30. A has m dollars, and B has y dollars. If A gives B 4 dollars, find the amount each will then have.

31. How many cents are in a dollars? In b dimes?

32. A man has a dollars, b dimes, and c cents. How many cents has he?

33. A room is x yards long, and y yards wide. How many square yards are there in the area of the floor?

34. Find the area of the floor of a room which is 2 yards longer and 3 yards wider than the one mentioned in Ex. 33.

35. The area of a rectangular field equals a square feet and its length equals b feet. Find (a) the width of the field. (b) The length of a fence surrounding the field.

36. The sum of two numbers is $m + 2n$, and one number equals y . Find the other number.

37. By what must 7 be multiplied to produce a product $x + y$?

38. What is the excess of $m + n$ over $5 + n$?

39. What is the cost of 6 apples at x cents each?

40. What is the cost of 1 apple, if x apples cost 7 cents?

41. What is the price of 3 apples, if x apples cost n cents?

42. What is the price of x apples, if one dozen cost n cents?

43. A man bought apples at the rate of m cents per dozen and pears at the rate of n cents per dozen. How many cents did he pay for x apples and y pears?

44. A man spent x dollars in buying books at n dollars each. How many books did he buy?

45. x years ago a man was 20 years old. How old is he now?

46. If A is $2x + 4$ years old, how many more years must he live to be 80 years old?

47. A cistern is filled by a pipe in a minutes. What fraction of the cistern will be filled by the pipe in one minute?

48. If a man walks 4 miles an hour, how many miles will he walk in n hours?

49. If a man walks r miles an hour, how many miles will he walk in t hours?

50. If a man walks 20 miles in t hours, how many miles does he walk each hour?

51. If a man walks at the rate of r miles per hour, how many hours will it take him to walk 20 miles?

52. How many miles does a train move in t hours at the rate of x miles per hour? How far does it move in y minutes?

53. A man traveled a miles on foot, b miles by boat, and the remainder by train. How many miles did he travel by train if the whole journey was 100 miles?

54. Find 5% of $100a$.

56. Find $x\%$ of 7.

55. Find 6% of x .

57. Find $x\%$ of a .

97. To express in algebraic symbols the sentence: " a exceeds b by as much as b exceeds 9," we have to consider that in this statement "exceed" means minus ($-$), and "by as much as" means equals ($=$). Hence we have

a exceeds b by as much as c exceeds 9.

$$a - b = c - 9.$$

Similarly, the difference of the squares of a and b increased by 80 equals the excess of a^3 over 80.

$$a^2 - b^2 +$$

$$80 = a^3 - 80.$$

Or, $(a^2 - b^2) + 80 = a^3 - 80.$

In many cases it is possible to translate a sentence word by word in algebraic symbols; in other cases the sentence has to be changed to obtain the symbols.

There are usually several different ways of expressing a symbolical statement in words, thus

$a - b = c$ may be expressed as follows:

The difference between a and b is c .

a exceeds b by c .

a is greater than b by c .

b is smaller than a by c .

The excess of a over b is c , etc.

EXERCISE 36

Express the following sentences as equations:

1. The double of a is 10.
2. One third of b is 17.
3. The double of a exceeds one third of b by c .
4. The difference of a and b increased by 19 is c .
5. Three times the sum of a and b exceeds c by as much as c exceeds 7.
6. Four times the difference of x and y increased by one fifth of c is equal to 9 times the product of b and c .
7. The product of the sum and the difference of a and b diminished by 90 is equal to the sum of the squares of a and b divided by 7.
8. Twenty subtracted from $2a$ gives the same result as 7 subtracted from a .
9. Nine is as much below a as 17 is above a .
10. x is 5% of 720.
11. x is 6% of a .
12. 50 is $x\%$ of 700.
13. 20 is 9% of $a + b$.
14. 70 is $x\%$ of m .

15. If A's age is $x + 20$, B's age is $4x + 12$, and C's age is $3x + 10$, express in algebraic symbols the following statements:

- (a) A is 5 years older than B.
- (b) The sum of A's age and B's age is 60.
- (c) A's age exceeds B's age by as much as B's age exceeds C's age.
- (d) C is three times as old as A.
- (e) In 5 years A will be as old as C is now.
- (f) Five years ago the sum of B's and C's ages was 40.
- (g) In 6 years A will be as old as C was 2 years ago.
- (h) x years ago B's age was 20.
- (i) In x years the sum of A's and B's ages will be 70.
- (j) One half of A's age plus one third of B's age equals 40.

16. If A, B, and C have respectively $2x$, $3x - 700$, and $x + 1200$ dollars, express in algebraic symbols:

- (a) A has 5 dollars more than B.
- (b) If A gains \$20 and B loses \$40, they have equal amounts.
- (c) If each man gains \$500, the sum of A's, B's, and C's money will be \$12,000.
- (d) A and B together have \$200 less than C.
- (e) If B pays to C \$100, they have equal amounts.
- (f) If A pays to B \$50, and C gains \$500, then A and C together have \$600 more than B.

17. A sum of money consists of x dollars, a second sum of $5x - 30$ dollars, a third sum of $2x + 1$ dollars. Express as equations:

- (a) 5% of the first sum equals \$90.
- (b) $a\%$ of the second sum equals \$20.
- (c) $x\%$ of the first sum equals 6% of the third sum.
- (d) $a\%$ of the first sum exceeds $b\%$ of the second sum by \$900.
- (e) 4% of the first plus 5% of the second plus 6% of the third sum equals \$8000.
- (f) $x\%$ of the first equals one tenth of the third sum.

18. Express the following equations in words; using for the letters x and y the words "a number" and "another number."

$$(a) \ x + 12 = 22.$$

$$(e) \ x + (3x - 7) = 2x + 19.$$

$$(b) \ x - 12 = 17.$$

$$(f) \ x - y = 2y - x.$$

$$(c) \ 3x - 7 = 2x.$$

$$(g) \ 3x - (x - 2) = 122.$$

$$(d) \ \frac{x}{3} + \frac{x}{2} = 119.$$

$$(h) \ \frac{x}{3} + \frac{y}{2} - 3(x - y) = 12(x + y).$$

19. If A, B, and C have respectively x , y , and z dollars, express the following equations as verbal statements:

$$(a) \ x = 2y.$$

$$(f) \ z - y = 200.$$

$$(b) \ x = y + z.$$

$$(g) \ \frac{x}{4} + \frac{y}{3} = z.$$

$$(c) \ 3x = 400.$$

$$(d) \ \frac{7}{100}x = 200.$$

$$(h) \ x + 300 = z + 100.$$

$$(e) \ .05x + .04y = 212.$$

$$(i) \ x - 250 = y + 100.$$

PROBLEMS LEADING TO SIMPLE EQUATIONS

98. The simplest kind of problems contain only one unknown number. In order to solve them, denote the unknown number by x (or another letter) and express the given sentence as an equation. The solution of the equation gives the value of the unknown number.

The equation can frequently be written by translating the sentence word by word into algebraical symbols; in fact the equation is the sentence written in algebraic shorthand.

Ex. 1. Three times a certain number exceeds 40 by as much as 40 exceeds the number. Find the number.

Let x = the number.

Write the sentence in algebraic symbols.

Three times a certain no. exceeds 40 by as much as 40 exceeds the no.

$$3 \times x - 40 = 40 - x$$

Or, $3x - 40 = 40 - x.$

Transposing, $3x + x = 40 + 40.$

Uniting, $4x = 80.$

$$x = 20, \text{ the required number.}$$

Check. $3x$ or 60 exceeds 40 by 20; 40 exceeds 20 by 20.

Ex. 2. In 15 years A will be three times as old as he was 5 years ago. Find A's present age.

Let x = A's present age.

The verbal statement (1) may be expressed in symbols (2).

(1) In 15 years A will be three times as old as he was 5 years ago.

$$(2) \quad x + 15 = 3 \times (x - 5)$$

Or, $x + 15 = 3(x - 5).$

Simplifying, $x + 15 = 3x - 15.$

Transposing, $x - 3x = -15 - 15.$

Uniting, $-2x = -30.$

Dividing, $x = 15.$

Check. In 15 years A will be 30; 5 years ago he was 10; but $30 = 3 \times 10.$

Ex. 3. To a quantity of water contained in a vessel, 56 gallons were added, and there was then in the vessel 8 times as much as at first. How many gallons did the vessel contain at first?

Let x = the number of gallons contained in the vessel at first.

The verbal statement expressed in letters gives $x + 56 = 8x.$

Transposing, $x - 8x = -56.$

Uniting, $-7x = -56.$

Dividing, $x = 8,$ the required number of gallons.

Check. If 56 gallons are added to 8 gallons, the result is 64 or 8×8 gallons.

NOTE. The student should note that x stands for the *number* of gallons, and similarly in other examples for *number* of dollars, *number* of yards, etc.

Ex. 4. 56 is what per cent of 120?

Let x = number of per cent, then the problem expressed in symbols would be

$$56 = \frac{x}{100} \cdot 120$$

or $\frac{4}{5}x = 56.$

Dividing, $x = 46\frac{2}{3}.$

Hence 56 is $46\frac{2}{3}\%$ of 120.

EXERCISE 37

1. What is the number which when subtracted from 40 will give the same result as when added to 14?
2. Find the number whose double increased by 4 equals 22.
3. What number added to three times itself gives a sum of 44?
4. Find the number whose double exceeds 7 by 5.
5. Four times a certain number diminished by 6 is equal to three times the number increased by 2. Find the number.
6. What number exceeds 6 by as much as three times the number exceeds 24?
7. What number is as much below 25 as four times the number is above 30?
8. If 47 be added to 9 times a certain number, the result will be 11 times the excess of the number over 1. Find the number.
9. Five times a certain number is greater by 4 than six times the difference of the number and 2. Find the number.
10. Thirty years hence A will be five times as old as he is now. Find his present age.
11. A man walks a certain distance, then travels three times as far by train, and then travels 30 miles by boat. If the whole journey is 77 miles, how far does he walk?
12. Twenty-eight years hence a man will be twice as old as he will be two years hence. How old is he now?
13. Fifteen years hence a man will be twice as old as he was 5 years ago. How old is he now?
14. To each of the numbers 1, 13, and 5, an unknown number is added. If the product of the first two sums is equal to the square of the last sum, what is the number?

15. A cistern is filled by a pipe, and the quantity let in after 8 minutes is 45 gallons more than the quantity let in after 5 minutes. Find the number of gallons let in per minute.

16. A train moving at a uniform rate runs in 5 hours 41 miles more than in 3 hours. How many miles per hour does it run?

17. A man is 33 years old, and his son is 12 years old. How many years ago was the father four times as old as the son?

18. A man is 36 years old, and his son is 11 years old. How many years hence will the father be twice as old as the son?

19. 60 is 5 % of what number?

20. 12 is what per cent of 22?

21. A has \$20, and B has \$30. How many dollars must B give to A to make A's money equal to 4 times B's money?

22. John and Henry have the same number of marbles. If John buys 20 marbles more and Henry loses 10, then John will have four times as many as Henry. How many has each?

23. A man bought two houses for the same price. He sold one at a profit of \$3000, and the other at a loss of \$1500, receiving twice as much for the first as for the last. How much did he pay for the houses?

24. A man wished to purchase a farm containing a certain number of acres. He found one farm which contained 20 acres too many, and another which lacked 15 acres of the required number. If the first farm contained twice as many acres as the second one, how many acres did he wish to buy?

25. If the sixth part of a number be added to 18, the result is the same as if three quarters of the number were subtracted from 29. Find the number.

99. If a problem contains two unknown quantities, two verbal statements must be given. In the simpler examples these two statements are given directly, while in the more complex problems they are only implied. We denote one of the unknown numbers (usually the smaller one) by x , and use one of the given verbal statements to express the other unknown number in terms of x . The other verbal statement, written in algebraic symbols, is the equation, which gives the value of x .

Ex. 1. One number exceeds another by 8, and their sum is 14. Find the numbers.

The problem consists of two statements:

I. One number exceeds the other one by 8.

II. The sum of the two numbers is 14.

Either statement may be used to express one unknown number in terms of the other, although in general the simpler one should be selected.

If we select the first one, and

Let x = the smaller number,

Then $x + 8$ = the greater number.

The second statement written in algebraic symbols produces the equation

$$x + (x + 8) = 14.$$

Simplifying,

$$x + x + 8 = 14.$$

Transposing,

$$x + x = 14 - 8$$

Uniting,

$$2x = 6.$$

Dividing,

$$x = 3, \text{ the smaller number.}$$

$$x + 8 = 11, \text{ the greater number.}$$

Another method for solving this problem is to express one unknown quantity in terms of the other by means of statement II; viz. the sum of the two numbers is 14.

Let

$$x = \text{the smaller number.}$$

Then,

$$14 - x = \text{the larger number.}$$

Statement I expressed in symbols is $(14 - x) - x = 8$, which leads of course to the same answer as the first method.

Ex. 2. A has three times as many marbles as B. If A gives 25 marbles to B, B will have twice as many as A.

The two statements are :

- I. A has three times as many marbles as B.
- II. If A gives B 25 marbles, B will have twice as many as A.

Use the simpler statement, viz. I, to express one unknown quantity in terms of the other.

Let $x = \text{B's number of marbles.}$

Then, $3x = \text{A's number of marbles.}$

To express statement II in algebraic symbols, consider that by the exchange A will lose, and B will gain.

Hence, $x + 25 = \text{B's number of marbles after the exchange.}$

$3x - 25 = \text{A's number of marbles after the exchange.}$

Therefore, $x + 25 = 2(3x - 25).$ (Statement II)

Simplifying, $x + 25 = 6x - 50.$

Transposing, $x - 6x = -25 - 50.$

Uniting, $-5x = -75.$

Dividing, $x = 15, \text{B's number of marbles.}$

$3x = 45, \text{A's number of marbles.}$

Check. $45 - 25 = 20, 15 + 25 = 40, \text{but } 40 = 2 \times 20.$

100. The numbers which appear in the equation should always be expressed in the same denomination. Never add the number of dollars to the number of cents, the number of yards to their price, etc.

Ex. 3. Eleven coins, consisting of half dollars and dimes, have a value of \$3.10. How many are there of each?

The two statements are :

- I. The number of coins is 11.
- II. The value of the half dollars and dimes is \$3.10.

Let, $x = \text{the number of dimes, then, from I,}$

$11 - x = \text{the number of half dollars.}$

Selecting the cent as the denomination (in order to avoid fractions) we express the statement II in algebraic symbols.

$$50(11 - x) + 10x = 310.$$

Simplifying, $550 - 50x + 10x = 310.$

Transposing, $-50x + 10x = -550 + 310.$

Uniting, $-40x = -240.$

Dividing, $x = 6$, the number of dimes.

$11 - x = 5$, the number of half dollars.

Check. 6 dimes = 60 cents, 5 half dollars = 250 cents, their sum is \$3.10.

EXERCISE 38

1. Two numbers differ by 33, and the greater is four times the smaller. Find the numbers.

2. Find two numbers whose sum is 72 and the greater of which equals five times the smaller.

3. The difference between two numbers is 8, and if 16 be added to the greater, the result will be three times the smaller. Find the numbers.

4. The difference between two numbers is 2, and the difference between their squares is 16. Find the numbers.

5. The sum of two numbers is 47, and their difference is 9. Find the numbers.

6. Divide 20 into two parts, one of which increased by 14 shall be equal to the other increased by 10.

7. One number is 5 less than three times another number. If the second number is subtracted from five times the first number, the result is 25. What are the numbers?

8. Divide 22 into two parts such that one part multiplied by 5 is equal to the other part diminished by 2.

9. Find two consecutive numbers whose sum is equal to 243.

10. Find two consecutive numbers, the difference of whose squares is equal to 27.

11. A's age is three times B's, and in 10 years A's age will be twice B's. Find their ages.

12. A and B divide a sum of money. A receives \$5 more than B, and five times B's money diminished by three times A's money equals \$25. How much does each receive?

13. The length of a rectangular field is three times its width, and a fence surrounding the field is 248 yards. Find the length and width of the field.

14. A is 27 years older than B, and B's age is as much below 20 as A's age is above 33. What are their ages?

15. Two vessels contain together 7 pints. If the smaller contained 2 pints more, it would contain half as much as the larger one. How many pints are there in each vessel?

16. On December 21, the night in St. Petersburg lasts 13 hours longer than the day. How many hours does the day last?

17. The difference of two numbers is 8, and their sum is five times the smaller. Find the two numbers.

18. A man built a house costing three times as much as the lot. If the house cost \$8000 more than the lot, what was the price of each?

19. A line 45 inches long is divided into two parts. Twice the larger part exceeds three times the smaller part by 30 inches. How many inches are there in each part?

20. A has \$16 less than B. If B gives \$20 to A, A will have five times as much as B. How many dollars has each?

21. A commenced business with three times as much capital as B. During the first year A lost \$600, B gained \$200, and A had then only twice as much as B. How much capital had each at first?

22. A man has \$6.25 in half dollars and quarters. He has three times as many quarters as half dollars. How many half dollars and quarters has he? (Ex. 3, \$100.)

23. The sum of \$3.50 is made up of 19 coins, which are either half dollars or dimes. How many are there of each?

24. John and Henry together have 50 marbles. If John had 6 marbles more, he would have three times as many as Henry. How many has each?

25. John and Henry together have 60¢. If John gives Henry 6¢ he will have 4 times as much as Henry. How many cents has each?

26. Henry bought 13 apples, some at the rate of 4¢ per apple, the rest at 3¢ per apple. How many did he buy of each kind if he paid 45¢ in all?

27. A and B together buy 200 lbs. of sugar. A takes 113 lbs. and B takes the remainder. If A uses $3\frac{1}{2}$ lbs. per day and B uses $2\frac{1}{3}$ lbs. per day, after how many days will they have equal quantities of sugar?

101. If a problem contains three unknown quantities, three verbal statements must be given. One of the unknown numbers is denoted by x , and the other two are expressed in terms of x by means of two of the verbal statements. The third verbal statement produces the equation. If four or more unknown quantities occur in the problem, the method is similar.

If it should be difficult to express the selected verbal statement directly in algebraical symbols, try to obtain it by a series of successive steps.

Ex. 1. A, B, and C together have \$80, and B has three times as much as A. If A and B each gave \$5 to C, then three times the sum of A's and B's money would exceed C's money by as much as A had originally.

The three statements are :

I. A, B, and C together have \$80.

II. B has three times as much as A.

III. If A and B each gave \$5 to C, then three times the sum of A's and B's money would exceed C's money by as much as A had originally.

Let x = the number of dollars A has.

According to II, $3x$ = the number of dollars B has,

and according to I, $80 - 4x$ = the number of dollars C has.

To express statement III by algebraical symbols, let us consider first the words "if A and B each gave \$5 to C."

$x - 5$ = number of dollars A had after giving \$5.

$3x - 5$ = number of dollars B had after giving \$5.

$90 - 4x$ = number of dollars C had after receiving \$10.

Expressing in symbols :

Three times the sum of A's and B's money exceeds C's money by A's

original amount.

$$3 \times (x - 5 + 3x - 5) - (90 - 4x) = x.$$

The solution gives $x = 8$, number of dollars A had.

$3x = 24$, number of dollars B had.

$80 - 4x = 48$, number of dollars C had.

Check. If A and B each gave \$5 to C, they would have 3, 19, and 58 respectively. $3(3 + 19)$ or 66 exceeds 58 by 8.

Ex. 2. A man spent \$1185 in buying horses, cows, and sheep, each horse costing \$90, each cow \$35, and each sheep \$15. The number of cows exceeded the number of horses by 4, and the number of sheep was twice as large as the number of horses and cows together. How many animals of each kind did he buy ?

The three statements are :

I. The total cost equals \$1185.

II. The number of cows exceeds the number of horses by 4.

III. The number of sheep is equal to twice the number of horses and cows together.

Let $x =$ the number of horses,
 then, according to II, $x + 4 =$ the number of cows,
 and, according to III,

$$2(2x + 4) \text{ or } 4x + 8 = \text{the number of sheep.}$$

Therefore, $90x =$ the number of dollars spent for horses.
 $35(x + 4) =$ the number of dollars spent for cows.
 and, $15(4x + 8) =$ the number of dollars spent for sheep.

Hence statement I may be written,

$$90x + 35(x + 4) + 15(4x + 8) = 1185.$$

Simplifying, $90x + 35x + 140 + 60x + 120 = 1185.$

Transposing, $90x + 35x + 60x = -140 - 120 + 1185.$

Uniting, $185x = 925.$

Dividing, $x = 5,$ number of horses.

$x + 4 = 9,$ number of cows.

$4x + 8 = 28,$ number of sheep.

Check. 5 horses, 9 cows, and 28 sheep would cost $5 \times 90 + 9 \times 35 + 28 \times 15$ or $450 + 315 + 420 = 1185$; $9 - 5 = 4$; $28 = 2(9 + 5).$

EXERCISE 39

1. Find three numbers such that the second is three times the first, the third is four times the first, and the difference between the third and the second is five.

2. Find three numbers such that the sum of the first two is 14, the third is twice the first, and the third exceeds the second by 4.

3. Find three numbers such that the second exceeds the first by 3, the third exceeds the first by 10, and the third is twice the first.

4. Find three numbers such that the sum of the first and second is 5, the sum of the first and last is 6, and the last is twice the first.

5. Find three numbers such that the sum of the first two is 8, the second exceeds the last by one, and the square of the last exceeds the square of the first by 7.

6. The difference between two numbers is 4; a third number is 5 less than the sum of the first two; and the sum of the first and third numbers is 14. What are the numbers?

7. Divide 20 into three parts such that the second part is twice the first, and the first part exceeds the last by 4.

8. A is three times as old as B, and C is five years younger than A. Three years ago the sum of their ages was 56 years. What are their ages?

9. A is five years older than B, and three years younger than C. Seventeen years ago C was twice as old as B. Find the age of each.

10. A man has 5 sons each three years older than the next younger. The age of the eldest three years hence will be three times the present age of the youngest. Find the age of each.

11. Divide 50 into three parts such that the first part is 8 more than the second, and the third increased by 34 is twice as large as the sum of the first and second parts.

12. A is five years older than B, and C is three times as old as B was five years ago. In five years C's age will be 10 times the difference between A's and B's ages. Find the age of each.

13. The three angles of any triangle are together equal to 180° . If the second angle of a triangle is 10° larger than the first, and the third is twice the sum of the first and second, what are the three angles?

14. There are 420 sheep in three flocks. The second contains twenty sheep more than the first, and the third twice as many as the first. How many sheep are there in each flock?

15. There are 500 sheep in three flocks. The second contains twice as many as the first, and if 10 be taken from the third and added to the first, the third will contain as many as the first and second together. How many sheep are there in each flock?

16. Find three consecutive numbers whose sum is 126.

17. Find four consecutive numbers such that the last is twice the first.

18. Find three consecutive numbers such that the difference of the squares of the third and first is 20.

19. A, B, and C divide a certain sum of money. A receives \$50 more than B. A and C together receive \$300, and B has \$150 less than C. How much does each receive?

20. Three boys, A, B, and C, divide a certain number of marbles, so that A and B together receive 22, A and C 25, and B and C 27. How many marbles does each receive?

21. A, B, and C have together 240 sheep. B has twenty more than A, and C has as many as A and B together. How many sheep has each?

22. B has twice as many acres as A, and C has three times as many acres as B. If A and C together have 1400 acres, how many acres has each?

23. A has one third as much money as B. C has \$25 more than A, and \$15 more than B. How much has each?

24. In a room there were twice as many women as children, and three more men than children. The number of men and women together was 9. How many children were there present?

25. A horse, carriage, and harness cost \$300. The horse costs \$10 less than the carriage and \$70 more than the harness; and the carriage and harness together cost \$180. Find the cost of the horse.

26. In 3 classes there are 120 pupils. The second class contains 10 more than the first, and the first and third together have 80 pupils. How many pupils are there in each class?

27. The sum of the four angles of any quadrilateral is 360° . If the second angle is twice as large as the first, the third twice as large as the second, and the fourth 30° larger than the third, find the value of each angle.

28. A has as many pennies as B has dollars, C has as many five-dollar bills as B has dollars, and together they have \$72.12. How much has each?

29. Three farms contain together 1280 acres. The first contains 200 acres more than $\frac{3}{8}$ of the second, and the third 10 acres less than $\frac{5}{8}$ of the second. How many acres does each farm contain?

30. The capacity of the first of three barrels is $\frac{5}{8}$ of that of the second, and the capacity of the second $\frac{8}{15}$ that of the third. If the contents of the third barrel be poured into the empty first barrel, there will be 10 gallons left. How many gallons does each barrel hold?

31. The surface of the earth consists of 5 zones, the torrid, two temperate, and two frigid zones. Each temperate zone is $\frac{1}{2}\frac{5}{3}$ of the torrid, each frigid is $\frac{7}{44}$ of each temperate, and the surface of the earth is approximately 200,000,000 square miles. How many square miles does each zone contain?

102. Arrangement of Problems. If the example contains quantities of 3 or 4 different kinds, such as length, width, and area, or time, speed, and distance, it is frequently advantageous to arrange the quantities in a systematic manner.

E.g. A and B start at the same hour from two towns 27 miles apart, B walks at the rate of 4 miles per hour, but stops 2 hours on the way, and A walks at the rate of 3 miles per hour without stopping. After how many hours will they meet and how many miles does A walk?

	TIME (in hours)	RATE (miles per hour)	DISTANCE (miles)
A	x	3	$3x$
B	$x - 2$	4	$4(x - 2)$

Explanation. First fill in all the numbers given directly, *i.e.* 3 and 4.

Let x = number of hours A walks, then $x - 2$ = number of hours B walks. Since in uniform motion the distance is always the product of rate and time, we obtain $3x$ and $4(x - 2)$ for the last column. But the statement "A and B walk from two towns 27 miles apart until they meet" means the sum of the distances walked by A and B equals 27 miles.

Hence $3x + 4(x - 2) = 27.$

Simplifying, $3x + 4x - 8 = 27.$

Uniting, $7x = 35.$

Dividing, $x = 5$, number of hours.

$3x = 15$, number of miles A walks.

This is, of course, not a new method for solving problems, but simply a convenient mode of arranging the solutions of some examples, which, however, may be solved without this arrangement.

Whenever various denominations occur repeatedly in the same connection, or when similar operations have to be performed repeatedly with several quantities, this arrangement will be found advantageous. Examples in which one quantity is found by multiplying the numerical values of two or more quantities belong to this group. In the following list the numerical values of the last column are equal to the products of the numerical values of the first two columns, provided care is taken in regard to the denomination.

Length	width	and area of rectangle
Rate of speed	time	distance covered
Number of persons	number of dollars each paid	number of dollars all paid
Number of yards	price per yard	total cost
Number of coins	value of each coin	total value of coins
Principal	rate per cent	interest

Ex. 1. The length of a rectangular field is twice its width. If the length were increased by 30 yards, and the width decreased by 10 yards, the area would be 100 square yards less. Find the dimensions of the field.

	LENGTH (yards)	WIDTH (yards)	AREA (square yards)
First field	$2x$	x	$2x^2$
Second field . . .	$2x + 30$	$x - 10$	$(2x + 30)(x - 10)$

The area would be decreased by 100 square yards gives

$$(2x + 30)(x - 10) = 2x^2 - 100.$$

Simplify, $2x^2 + 10x - 300 = 2x^2 - 100.$

Cancel $2x^2$ and transpose, $10x = 200.$

$$x = 20.$$

$$2x = 40.$$

The field is 40 yards long and 20 yards wide.

Check. The original field has an area $40 \times 20 = 800$, the second field 70×10 or 700. But $700 = 800 - 100$.

Ex. 2. A certain sum invested at 5% brings the same interest as a sum \$200 larger at 4%. What is the capital?

PRINCIPAL (No. of dollars)	RATE %	INTEREST (No. of dollars)
x	.05	$.05x$
$x + 200$.04	$.04(x + 200)$

Therefore $.05x = .04(x + 200).$

Simplify, $.05x = .04x + 8.$

Transposing and uniting, $.01x = 8.$

Multiplying, $x = 800$; \$800 = required sum.

Check. $\$800 \times .05 = \40 ; $\$1000 \times .04 = \$40.$

EXERCISE 40

1. A rectangular field is 20 yards, and another 25 yards wide. The second is 10 yards longer than the first, and the sum of their areas is equal to 1600 square yards. Find the dimensions of each. ✓

2. A rectangular field is 14 yards longer than it is wide. If its length were increased by 10 yards, and its width decreased by 4 yards, the area would remain the same. Find the dimensions of the field.

3. A rectangular field is twice as long as it is wide. If it were 50 feet shorter and 20 feet wider, it would contain 2000 square feet less. Find the dimensions of the field.

4. A certain sum invested at 4% brings the same interest as a sum \$300 larger invested at 3%. Find the first sum.

5. A sum invested at 5%, and a second sum, twice as large, invested at 4%, together bring \$52 interest. What are the two sums?

6. An investment of \$2500 brings a yearly interest of \$114. A part of the capital is invested at 4%, and the remainder at 5%. How many dollars are invested at 4%?

7. A bought 12 oranges for a certain sum. If each orange had cost one cent more, he would have received 10 oranges for the same money. What was the price of each orange?

8. Six persons bought an automobile, but as two of them were unable to pay their share, each of the others had to pay \$40 more. Find the share of each, and the cost of the automobile.

9. Ten yards of silk and 20 yards of cloth cost together \$35. If the silk cost three times as much per yard as the cloth, how much did each cost per yard?

10. Twenty yards of silk and 30 yards of cloth cost together \$85. If the silk cost 50¢ more per yard than the cloth, what was the price of each per yard?

11. A man bought 7 lbs. of coffee for \$1.79. For a part he paid 24¢ per lb. and for the rest he paid 35¢ per lb. How many pounds of each kind did he buy?

12. Sixteen persons subscribed \$138. Six of them paid equal amounts, and the remaining ones paid each one dollar more. Find the share of each man.

13. Twenty men subscribed equal amounts to raise a certain sum of money, but four men failed to pay their shares, and in order to raise the required sum each of the remaining men had to pay one dollar more. How much did each man subscribe?

14. A cistern is filled in a certain time by a pipe which lets in 20 gallons per minute. Another pipe letting in 25 gallons per minute fills the cistern in one minute less. In how many minutes does the first pipe fill the cistern?

15. A cistern is filled in a certain time by a pipe letting in 21 gallons per minute. If another pipe, which lets in 14 gallons per minute, is opened 3 minutes longer than the first, 6 gallons less than in the first case will be poured in. In how many minutes does the first pipe fill the cistern?

16. A sets out walking at the rate of 3 miles per hour, and three hours later B follows on horseback traveling at the rate of 6 miles per hour. After how many hours will B overtake A, and how far will each then have traveled?

17. A and B set out walking at the same time in the same direction, but A has a start of 3 miles. If A walks at the rate of $2\frac{1}{2}$ miles per hour, and B at the rate of 3 miles per hour, how far must B walk before he overtakes A?

18. A sets out walking at the rate of 3 miles per hour, and one hour later B starts from the same point traveling by coach in the opposite direction at the rate of 6 miles per hour. After how many hours will they be 27 miles apart?

19. A and B start walking at the same hour from two towns $17\frac{1}{2}$ miles apart, and walk toward each other. If A walks at the rate of 3 miles per hour, and B at the rate of 4 miles per hour, after how many hours do they meet and how many miles does A walk?

20. The distance from New York to Albany is 142 miles. If a train starts at Albany and travels toward New York at the rate of 40 miles per hour without stopping, and another train

starts at the same time from New York traveling at the rate of 42 miles an hour, how many miles from New York will they meet?

21. Two men start at 12 o'clock from two towns 17 miles apart, and travel toward each other. One walks at the rate of 3 miles per hour, but rests one hour on the way; the other travels at the rate of 4 miles per hour and rests 3 hours. At what hour do they meet?

22. A and B start from two towns 20 miles apart and travel toward each other. A starts at 1 P.M., B starts at 2 P.M., and they meet at 5 P.M. If B travels one mile per hour faster than A, find the number of miles each travels per hour.

23. A picture which is 2 inches longer than wide is surrounded by a frame 1 inch wide. If the area of the frame is 40 square inches, what are the dimensions of the picture?

MISCELLANEOUS PROBLEMS

24. The formula which transforms Fahrenheit readings of a thermometer into Centigrade readings is $C = \frac{5}{9}(F - 32)$.

If $C = 40^\circ$, find the value of F .

25. Change the following readings to Fahrenheit readings: (a) 0° C, (b) 100° C, (c) 50° C, (d) -12° C.

26. At what temperature do the Centigrade scale and Fahrenheit scale indicate equal numbers?

27. The formula for the distance which a falling body passes over in t seconds is $S = \frac{1}{2}gt^2$. (Ex. 7, p. 16.)

If $S = 240$ ft. and $t = 4$ seconds, find the value of g .

28. The formula for compound interest is

$$I = p \left(1 + \frac{r}{100} \right)^n - p.$$

(For the meaning of the letters see Ex. 4, p. 16.)

Find the principal that will bring \$662 interest in two years at 10% compound interest.

29. A number increased by 7 gives the same result as the number multiplied by 7. What is the number?

30. If a number be added to 3, the sum multiplied by 3, the product diminished by 20, the difference multiplied by 6, and the product diminished by 55, the result will be 5. Find the number.

31. A has as many dollars as B has cents. If A should give B \$6.93, B would have as many dollars as A has cents. How much money has each?

32. A man made as much money as he had and \$100. He made as much money as he then had and \$200; again he made as much as he then had and \$300, and found that he had finally \$3100. How many dollars had he at first?

33. A man met some beggars, and after giving each 4¢ had 9¢ left. He found that he lacked 7¢ to be able to give each beggar 6¢. How many beggars were there?

34. A mason working 8 hours a day, in the course of a week, builds a number of cubic meters which exceeds 43 as much as 43 exceeds the number of cubic meters which he would build working $7\frac{1}{2}$ hours a day. How many cubic meters does he build per hour?

35. A has \$6 more than B and gives to B as much as B has. Then B gives to A as much as A then has, and once more A gives to B as much as B then has; and finds that A has now as much as B. How many dollars has each at first?

36. A boy has the same number of sisters as brothers, while his sister has $1\frac{1}{2}$ times as many brothers as sisters. How many sons and daughters are there in the family?

CHAPTER VI

FACTORING

103. An expression is **rational with respect to a letter**, if, after simplifying, it contains no indicated root of this letter; **irrational**, if it does contain some indicated root of this letter.

$a^2 - \frac{1}{a} + \sqrt{b}$ is rational with respect to a , and irrational with respect to b .

104. An expression is **integral** with respect to a letter, if this letter does not occur in any denominator.

$\frac{a^2}{b} + ab + b^2$ is integral with respect to a , but fractional with respect to b .

105. An expression is **integral and rational**, if it is integral and rational with respect to all letters contained in it; as,

$$a^2 + 2ab + 4c^2.$$

106. The **factors** of an algebraic expression are the quantities which multiplied together will give the expression.

In the present chapter only integral and rational expressions are considered factors.

Although $\sqrt{a^2 - b^2} \times \sqrt{a^2 - b^2} = a^2 - b^2$, we shall not, at this stage of the work, consider $\sqrt{a^2 - b^2}$ a factor of $a^2 - b^2$.

107. A factor is said to be **prime**, if it contains no other factors (except itself and unity); otherwise it is **composite**.

The prime factors of $10a^3b$ are 2, 5, a , a , a , b .

108. Factoring is the process of separating an expression into its factors. An expression is factored if written in the form of a product.

$(x^2 - 4x + 3)$ is factored if written in the form $(x - 3)(x - 1)$. It would not be factored if written $x(x - 4) + 3$, for this result is a sum, and not a product.

109. The factors of a monomial can be obtained by inspection.

The prime factors of $12x^3y^2$ are 3, 2, 2, x , x , x , y , y .

110. Since factoring is the inverse of multiplication, it follows that every method of multiplication will produce a method of factoring.

E.g. since $(a + b)(a - b) = a^2 - b^2$, it follows that $a^2 - b^2$ can be factored, or that $a^2 - b^2 = (a + b)(a - b)$.

111. Factoring examples may be checked by multiplication or by numerical substitution.

TYPE I. POLYNOMIALS ALL OF WHOSE TERMS CONTAIN A COMMON FACTOR

$$mx + my + mz = m(x + y + z). \quad (\S 55.)$$

112. Ex. 1. Factor $6x^3y^2 - 9x^2y^3 + 12xy^4$.

The greatest factor common to all terms is $3xy^2$. Divide

$$6x^3y^2 - 9x^2y^3 + 12xy^4 \text{ by } 3xy^2,$$

and the quotient is $2x^2 - 3xy + 4y^2$.

But, dividend = divisor \times quotient.

$$\text{Hence} \quad 6x^3y^2 - 9x^2y^3 + 12xy^4 = 3xy^2(2x^2 - 3xy + 4y^2).$$

Ex. 2. Factor

$$14a^4b^2c^2 - 21a^2b^4c^2 + 7a^2b^2c^2 = 7a^2b^2c^2(2a^2 - 3b^2 + 1).$$

EXERCISE 41

Factor the following expressions :

1. $15 abx - 9 b^2x$.
2. $9 a^3 - 6 a^2$.
3. $16 x^3 - 4 x^2$.
4. $14 acd - 7 cd + 21 c^2d^2$.
5. $3 a^3 - 6 a^2 + 9 a$.
6. $8 p^4y + 2 p^3y^2 - 6 p^2y^4$.
7. $5 x^2y^2 - 15 xy + 20 xyz$.
8. $7 m - 7 mn - 7 p$.
9. $4 a^3b - 5 a^2c^2 + 6 ad^3$.
10. $17 m^3n^3 - 51 m^2n^2 + 85 mn$.
11. $15 a^2b^2x - 9 b^3y + 12 b^4$.
12. $9 x^9y^8z^7 - 6 x^8y^8z^8 + 3 x^7y^8z^9$.
13. $14 x^3y^4 - 21 x^3y^3z + 49 x^3y^2z^2$.
14. $12 m^3n^3 - 18 m^2n^2 - 24 m^4n^4$.
15. $11 p^4q^2 - 33 p^3q^3 + 11 p^2q^4$.
16. $x^4 + x^3 - x^2 + x$.
17. $39 a^3b^4c^5 - 26 a^4b^5c^6 + 13 a^5b^6c^7$.
18. $51 m^6y^6 - 34 m^5y^5 + 12 m^4y^4$.
19. $2 x^4 - 4 x^3y + 6 x^2y^2 + 8 y^3$.
20. $4 x^4y^2 - 28 x^3y^3 + 40 x^2y^4 - 48 xy^5$.
21. $x(a+b) + y(a+b)$.
22. $3 x^2(m+n) - 2 y^3(m+n)$.
23. $6 a^2b^2(p+q) - 4 ab^4(p+q) - (p+q)$.
24. $4 x^2(x-y) - 7 z^2(x-y)$.
25. $4 x^n - 12 x^{n+1} - 6 x^{n+2}$.
26. $6 a^{2n}b^n - 3 a^n b^{2n}$.
27. $x^2(x-3) - 3 x(x-3)$.
28. $3 x^2(x+9) - (x+9)$.
29. $am + bm + an + bn$.

TYPE II. QUADRATIC TRINOMIALS OF THE FORM

$$x^2 + px + q.$$

113. In multiplying two binomials containing a common term, *e.g.* $(x-3)$ and $(x+5)$, we had to add -3 and 5 to obtain the coefficient of x , and to multiply -3 and 5 to obtain the term which does not contain x or $(x-3)(x+5) = x^2 + 2x - 15$.

In factoring $x^2 + 2x - 15$ we have, obviously, to find two numbers whose product is -15 and whose sum is $+2$.

Or, in general, in factoring a trinomial of the form $x^2 + px + q$, we have to find two numbers m and n whose sum is p , and whose product is q ; and if such numbers can be found, the factored expression is $(x + m)(x + n)$.

Ex. 1. Factor $x^2 - 4x - 77$.

We may consider -77 as the product of $-1 \cdot 77$, or $-7 \cdot 11$, or $-11 \cdot 7$, or $-77 \cdot 1$, but of these only -11 and 7 have a sum equal to -4 .

Hence
$$x^2 - 4x - 77 = (x - 11)(x + 7).$$

Since a number can be represented in an infinite number of ways as the sum of two numbers, but only in a limited number of ways as a product of two numbers, it is advisable to consider the factors of q first. If q is positive, the two numbers have both the same sign as p . If q is negative, the two numbers have opposite signs, and the greater one has the same sign as p .

Not every trinomial of this type, however, can be factored.

Ex. 2. Factor $a^2 - 11a + 30$.

The two numbers whose product is 30 and whose sum is -11 are -5 and -6 .

Therefore
$$a^2 - 11a + 30 = (a - 5)(a - 6).$$

Check. If $a=1$, $a^2 - 11a + 30 = 20$, and $(a-5)(a-6) = -4 \cdot -5 = 20$.

Ex. 3. Factor $x^2 + 10ax - 11a^2$.

The numbers whose product is $-11a^2$ and whose sum is $10a$ are $11a$ and $-a$.

Hence
$$x^2 + 10ax - 11a^2 = (x + 11a)(x - a).$$

Ex. 4. Factor $x^6 - 7x^3y^3 + 12y^6$.

The two numbers whose product is equal to $12y^6$ and whose sum equals $-7y^3$ are $-4y^3$ and $-3y^3$. Hence $x^6 - 7x^3y^3 + 12y^6 = (x^3 - 3y^3)(x^3 - 4y^3)$.

Ex. 5. Factor $1 - 3a - 10a^2$.

This expression is a special form of the general type, obtained by letting $x = 1$.

Hence
$$1 - 3a - 10a^2 = (1 - 5a)(1 + 2a).$$

114. In solving any factoring example, the student should first determine whether all terms contain a common monomial factor.

EXERCISE 42

Factor the following expressions :

1. $x^2 - 5x + 6$.
2. $x^2 + 5x + 6$.
3. $a^2 - 3a + 2$.
4. $a^2 + 7a + 12$.
5. $x^2 - 4x - 21$.
6. $m^2 + 4m - 21$.
7. $q^2 + 5q - 14$.
8. $y^2 - 7y - 18$.
9. $y^2 - 8y + 15$.
10. $x^2 - 5x - 14$.
11. $x^2 + 2x + 1$.
12. $m^2 - 14m + 33$.
13. $x^2 - 3x - 4$.
14. $y^2 - 38y + 37$.
15. $y^2 - 36y - 37$.
16. $m^2 - 19m + 48$.
17. $b^2 - 14b - 51$.
18. $x^2 - 32x + 175$.
19. $x^2 - 8x + 16$.
20. $x^2 - 17ax + 30a^2$.
21. $a^2 - 12ab - 13b^2$.
22. $m^2 + 15mn - 34n^2$.
23. $p^2 - 7pqr + 12q^2r^2$.
24. $m^2 + 28mn + 187n^2$.
25. $a^2b^2 + 2ab - 35$.
26. $x^2y^2z^2 - 19xyz + 48$.
27. $a^4 - 4a^2 - 21$.
28. $x^6 + 17x^3 + 60$.
29. $a^4 - 11a^2b^2 + 24b^4$.
30. $a^4b^4 - 13a^2b^2c^2 - 30c^4$.
31. $9m + m^2 + 20$.
32. $a^2 + 2b^2 - 3ab$.
33. $7 - 8m + m^2$.
34. $3m - 4 + m^2$.
35. $1 - 7m + 12m^2$.
36. $m^2 - 25$.
37. $x^3 - 5x^2 + 6x$.
38. $3a^2 - 36a + 33$.
39. $3m^3 - 15m^2 - 18m$.
40. $5y^3 - 105y - 20y^2$.
41. $6ab^3 + a^3b - 5a^2b^2$.
42. $6x^3 + 2x^5 - 36x$.
43. $98xy^2 - 28x^2y + 2x^3$.
44. $x^{2m} - 7x^m + 12$.
45. $m^{x+2} - 9m^{x+1}n + 20m^xn^2$.
46. $(a+b)^2 - 7(a+b) - 18$.
47. $(a+b)^3 - 12(a+b)^2 + 20(a+b)$.
48. $a^2 - 3ab$.

TYPE III. QUADRATIC TRINOMIALS OF THE FORM

$$px^2 + qx + r.$$

115. According to § 66,

$$(4x + 3)(5x - 2) = 20x^2 + 7x - 6.$$

$20x^2$ is the product of $4x$ and $5x$.

-6 is the product of $+3$ and -2 .

$+7x$ is the sum of the cross products.

Hence in factoring $6x^2 - 13x + 5$, we have to find two binomials whose corresponding terms are similar, such that

The first two terms are factors of $6x^2$.

The last two terms are factors of 5 ,

and the sum of the cross products equals $-13x$.

By actual trial we find which of the factors of $6x^2$ and 5 give the correct sum of cross products.

If we consider that the factors of $+5$ must have like signs, and that they must be negative, as $-13x$ is negative, all possible combinations are contained in the following :

$$\begin{array}{r} 6x-1 \\ \times \\ x-5 \\ \hline -31x \end{array}$$

$$\begin{array}{r} 6x-5 \\ \times \\ x-1 \\ \hline -11x \end{array}$$

$$\begin{array}{r} 3x-1 \\ \times \\ 2x-5 \\ \hline -17x \end{array}$$

$$\begin{array}{r} 3x-5 \\ \times \\ 2x-1 \\ \hline -13x \end{array}$$

Evidently the last combination is the correct one, or

$$6x^2 - 13x + 5 = (3x - 5)(2x - 1).$$

116. In actual work it is not always necessary to write down all possible combinations, and after a little practice the student should be able to find the proper factors of simple trinomials at the first trial. The work may be shortened by the following considerations :

1. If p is positive, only positive factors of p need be considered.
2. If p and r are positive, the second terms of the factors have the same sign as q .

3. If p is positive, and r is negative, then the second terms of the factors have opposite signs.

If a combination should give a sum of cross products, which has the same absolute value as the term qx , but the opposite sign, exchange the signs of the second terms of the factors.

4. If $px^2 + qx + r$ does not contain any monomial factor, none of the binomial factors can contain a monomial factor.

Ex. 1. Factor $3x^2 - 83x + 54$.

The factors of the first term consist of one pair only, viz. $3x$ and x , and the signs of the second terms are minus. 54 may be considered the product of the following combinations of numbers: 1×54 , 2×27 , 3×18 , 6×9 , 9×6 , 18×3 , 27×2 , 54×1 . Since the first term of the first factor ($3x$) contains a 3, we have to reject every combination of factors of 54, whose first factor contains a 3. Hence only 1×54 and 2×27 need be considered.

$$\begin{array}{r} 3x - 1 \\ \quad \times \\ x - 54 \\ \hline -163x \end{array}$$

$$\begin{array}{r} 3x - 2 \\ \quad \times \\ x - 27 \\ \hline -83x \end{array}$$

Therefore $3x^2 - 83x + 54 = (3x - 2)(x - 27)$.

Ex. 2. Factor $9x^2 + 20x - 21$.

$9x^2 = 9x \cdot x$ or $3x \cdot 3x$, but the second combination has to be rejected, as 21 contains a 3, and as consequently one of the resulting binomial factors would contain the monomial factor 3. Hence $9x \cdot x$ has to be selected.

The last term 21 may be factored as follows: 1×21 , 3×7 , 7×3 , and 21×1 . According to (4), only 1×21 and 7×3 need to be tried.

$$\begin{array}{r} 9x + 1 \\ \quad \times \\ x - 21 \\ \hline -188x \end{array}$$

$$\begin{array}{r} 9x + 7 \\ \quad \times \\ x - 3 \\ \hline -20x \end{array}$$

The second combination produces the absolute value of the middle term but the wrong sign, hence the factors are

$$(9x - 7)(x + 3).$$

117. The type $px^2 + qx + r$ is the most important of the trinomial types, since all others (II, IV) are special cases of it. In all examples of this type, the expressions should be arranged according to the ascending or the descending powers of some letter, and the monomial factors should be removed.

EXERCISE 43

Factor the following expressions :

- | | |
|---------------------------|-----------------------------------|
| 1. $3x^2 - x - 2.$ | 21. $2a^2 - 3ab - 2b^2.$ |
| 2. $5a^2 - 9a - 2.$ | 22. $4a^2 - a - 14.$ |
| 3. $3x^2 - 10x + 3.$ | 23. $60a^2 - 59ab - 20b^2.$ |
| 4. $4x^2 + 7x - 2.$ | 24. $12x^4 - 23x^2 + 10.$ |
| 5. $3a^2 - 5a + 2.$ | 25. $8a^6 - 38a^3 + 35.$ |
| 6. $2x^2 - 9x + 4.$ | 26. $2 - 5a^2 + 3a^4.$ |
| 7. $9x^2 - 26x - 3.$ | 27. $3 - x - 2x^2.$ |
| 8. $4x^2 - 8x + 3.$ | 28. $6 - x - 2x^2.$ |
| 9. $4x^2 - 11x - 3.$ | 29. $12 - 2x^2 - 5x.$ |
| 10. $6b^2 + b - 12.$ | 30. $12 - x^2 - x.$ |
| 11. $4x^2 - 5x - 6.$ | 31. $-5x - x^2 + 6.$ |
| 12. $6a^2 - 19a + 10.$ | 32. $4x^2 + 10xy + 4y^2.$ |
| 13. $9m^2 - 17m - 2.$ | 33. $8x^2y^2 + 22xy^2 - 6y^2.$ |
| 14. $5x^2 + 26x + 5.$ | 34. $24a^3 + 42a^2b - 45ab^2.$ |
| 15. $6a^2 - 17a + 12.$ | 35. $30x^2y + 95x^2y^2 - 35xy^3.$ |
| 16. $4a^2 - 4ab - 3b^2.$ | 36. $3x^{n+2} + 5x^{n+1} + 2x^n.$ |
| 17. $4 + 13x + 3x^2.$ | 37. $2(a+b)^2 + 11(a+b) + 5.$ |
| 18. $6x^2 - 7xy - 3y^2.$ | 38. $4(x+y)^2 - 8(x+y) + 3.$ |
| 19. $2a^2 + 13ab + 6b^2.$ | 39. $3x^{2n} - x^n - 2.$ |
| 20. $15a^2 - 77a + 10.$ | 40. $ax^2 + (a+b)x + b.$ |

TYPE IV. THE SQUARE OF A BINOMIAL

$$x^2 \pm 2xy + y^2.$$

118. Expressions of this form are special cases of the preceding type, and may be factored according to the method used for that type. In most cases, however, it is more convenient to factor them according to § 65.

$$x^2 + 2xy + y^2 = (x + y)^2.$$

$$x^2 - 2xy + y^2 = (x - y)^2.$$

A trinomial belongs to this type, *i.e.* it is a perfect square, when two of its terms are perfect squares, and the remaining term is equal to twice the product of the square roots of these terms.

The student should note that a term, in order to be a perfect square, must have a positive sign.

$16x^2 - 24xy + 9y^2$ is a perfect square, for $2\sqrt{16x^2} \times \sqrt{9y^2} = 24xy$.

Evidently $16x^2 - 24xy + 9y^2 = (4x - 3y)^2$.

To factor a trinomial which is a perfect square, connect the square roots of the terms which are squares by the sign of the remaining term, and indicate the square of the resulting binomial.

EXERCISE 44

Determine whether the following expressions are perfect squares or not, and factor whenever possible:

1. $a^2 + 2ab + b^2.$

7. $p^2 + 8p + 16.$

2. $a^2 - 2ab - b^2.$

8. $1 - 4m + 4m^2.$

3. $c^2 + 2cd - d^2.$

9. $9 + 6p + p^2.$

4. $x^2 + 4x + 4.$

10. $4 - 6p + p^2.$

5. $x^2 - 6x + 9.$

11. $a^2 - 14a + 49.$

6. $x^2 - 2x + 4.$

12. $a^2 + 18a + 81.$

- | | |
|--------------------------------------|---------------------------------------|
| 13. $a^2 + 6a - 9$. | 25. $-a^2 + 2ab - b^2$. |
| 14. $a^2 - 6ab + 9b^4$. | 26. $168y^2 + x^2 - 26xy$. |
| 15. $4a^2 - 12ab + 9b^2$. | 27. $a^2 + 2ab$. |
| 16. $9m^2n^2 + 42mn + 49$. | 28. $+12x^n + 36x^{2n} + 1$. |
| 17. $4x^2y^2 - 20xyz + 25z^2$. | 29. $2m^3n^3 - m^6 - n^6$. |
| 18. $4x^4 - 15x^2y^2z^2 + 9y^4z^4$. | 30. $36x^2 + 6x^3 + 54x$. |
| 19. $36a^4 - 60a^2b^2 + 25b^4$. | 31. $a^{2n} - 2a^nb^m + b^{2m}$. |
| 20. $225x^6 - 30x^3 + 1$. | 32. $2^{2n} - 6 \cdot 2^n + 9$. |
| 21. $4a^2 - 8ab + 4b^2$. | 33. $98x^4y^2 - 56x^3y^3 + 8x^2y^4$. |
| 22. $m^3 + 2m^2 + m$. | 34. $25x^2y^2 - 101xy^2 + 4y^2$. |
| 23. $m^4 - 12m^2b^2 - 36b^4$. | 35. $4a^4 - 52a^3 + 144a^2$. |
| 24. $2a^2 + 12ab - 18b^2$. | 36. $(a+b)^2 - 24(a+b) + 144$. |
| 37. $(a-b)^2 - 2x(a-b) + x^2$. | |

TYPE V. THE DIFFERENCE OF TWO SQUARES

$$x^2 - y^2.$$

119. According to § 65,

$$a^2 - b^2 = (a+b)(a-b),$$

i.e. *the difference of the squares of two numbers is equal to the product of the sum and the difference of the two numbers.*

Ex. 1. $4x^6y^8 - 9z^6 = (2x^3y^4 + 3z^3)(2x^3y^4 - 3z^3).$

Ex. 2. $16a^8 - 64b^{10} = 16(a^8 - 4b^{10})$
 $= 16(a^4 + 2b^5)(a^4 - 2b^5).$

Ex. 3. $a^4 - b^4 = (a^2 + b^2)(a^2 - b^2)$
 $= (a^2 + b^2)(a+b)(a-b).$

NOTE. $a^2 + b^2$ is prime.

EXERCISE 45

Resolve into prime factors:

- | | |
|----------------------------|-------------------------------|
| 1. $m^2 - n^2$. | 16. $4a^{13} - 9ab^{12}$. |
| 2. $p^2 - q^2$. | 17. $7x^{16} - 7y^{16}$. |
| 3. $a^2 - 9$. | 18. $16x^3 - 196x$. |
| 4. $16 - b^2$. | 19. $9a^{10} - 81$. |
| 5. $4a^2 - 1$. | 20. $100 - 900a^{100}$. |
| 6. $1 - 25a^2b^2$. | 21. $a^4 - a^8$. |
| 7. $a^2b^2 - 25c^2$. | 22. $144a^2 - a^4$. |
| 8. $a^2b^2 - 36b^2$. | 23. $9x^2y^2z^2 - 9x^2y^2$. |
| 9. $49a^4b^4 - 16c^4$. | 24. $12a^8 - 3b^2c^2$. |
| 10. $36a^8 - 25b^{18}$. | 25. $a^{2m} - 1$. |
| 11. $169c^4 - 121d^6e^6$. | 26. $64a^{4m} - 9b^{2m}$. |
| 12. $225a^3 - 144ab^2$. | 27. $3^{2m} - 2^{2n}$. |
| 13. $a^6 - b^6$. | 28. $25a^{2+m} - 225a^mb^2$. |
| 14. $a^4 - b^4$. | 29. $10000 - 1$. |
| 15. $a^8 - b^8$. | 30. 99.91 . |

120. One or both terms are squares of polynomials.

Ex. 1. Factor $a^2 - (c + d)^2$.

$$a^2 - (c + d)^2 = (a + c + d)(a - c - d).$$

Ex. 2. Resolve into prime factors and simplify

$$(4a + 3b)^2 - (2a - 5b)^2.$$

$$\begin{aligned} (4a + 3b)^2 - (2a - 5b)^2 &= [(4a + 3b) + (2a - 5b)][(4a + 3b) - (2a - 5b)] \\ &= [4a + 3b + 2a - 5b][4a + 3b - 2a + 5b] \\ &= (6a - 2b)(2a + 8b) \\ &= 2(3a - b) \cdot 2(a + 4b) \\ &= 4(3a - b)(a + 4b). \end{aligned}$$

EXERCISE 46

Resolve into prime factors :

- | | |
|-------------------------|---------------------------------|
| 1. $(a+b)^2 - c^2$. | 7. $(a^3 - 2a^2)^2 - (a-b)^2$. |
| 2. $(x-y)^2 - z^2$. | 8. $(4a+b)^2 - (3x-y)^2$. |
| 3. $(a+b)^2 - 9c^2$. | 9. $1 - (3a-5b)^2$. |
| 4. $a^2 - (b+c)^2$. | 10. $3 - 3(a+b)^2$. |
| 5. $4a^2 - (b-c)^2$. | 11. $(a+b+c)^2 - (x+y-z)^2$. |
| 6. $(a+3b)^2 - 16c^2$. | 12. $(a+2b+3c+d)^2 - (e+f)^2$. |

Resolve into factors and simplify :

- | | |
|---------------------------|--------------------------------|
| 13. $(a+b)^2 - a^2$. | 17. $(3a+5)^2 - (2a-1)^2$. |
| 14. $a^2 - (a-b)^2$. | 18. $(3a^2 - a)^2 - (a+b)^2$. |
| 15. $(a+2b)^2 - 9a^2$. | 19. $(4a+6b)^2 - (a-7b)^2$. |
| 16. $25a^2 - (2a-6b)^2$. | 20. $(a+b+c)^2 - (a+b)^2$. |

TYPE VI. THE SUM OR DIFFERENCE OF TWO CUBES

$$x^3 + y^3, \text{ and } x^3 - y^3.$$

121. According to § 79 :

$$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$$

$$x^3 - y^3 = (x-y)(x^2 + xy + y^2).$$

Ex. 1. Factor $8x^6 - 1$.

$$\begin{aligned} 8x^6 - 1 &= (2x^2)^3 - (1)^3 \\ &= (2x^2 - 1)(4x^4 + 2x^2 + 1). \end{aligned}$$

Ex. 2. Factor $8a^9 + 27b^6$.

$$\begin{aligned} 8a^9 + 27b^6 &= (2a^3)^3 + (3b^2)^3 \\ &= (2a^3 + 3b^2)(4a^6 - 6a^3b^2 + 9b^4). \end{aligned}$$

After a little practice the student may omit the intermediate step and write the factors at once.

Ex. 3. $125a^6b^{12} - 343c^9$

$$= (5a^2b^4 - 7c^3)(25a^4b^8 + 35a^2b^4c^3 + 49c^6).$$

EXERCISE 47

Factor the following:

- | | | |
|---------------------|---------------------------|-------------------------------|
| 1. $a^3 - b^3$. | 10. $125 a^3 + b^3$. | 19. $3 x^3 y^3 - 81 x^{12}$. |
| 2. $a^3 + b^3$. | 11. $8 a^3 - 27 b^3$. | 20. $x^{81} - 1331$. |
| 3. $a^3 - 1$. | 12. $216 a^3 + 125 b^3$. | 21. $x^3 - 2^{3m}$. |
| 4. $a^3 + 1$. | 13. $1000 - x^{27}$. | 22. 1001. |
| 5. $1 + a^6$. | 14. $216 a^3 - b^6 c^6$. | 23. 1,000,001. |
| 6. $8 - a^3$. | 15. $x^6 - 27 x^3 y^3$. | 24. 1,000,027. |
| 7. $8 a^3 + 1$. | 16. $2 x^3 + 54 y^3$. | 25. 64,001. |
| 8. $27 b^3 - 1$. | 17. $a^{12} + b^{12}$. | 26. 64,000,001. |
| 9. $64 a^3 - b^3$. | 18. $a^7 + 27 ab^6$. | 27. 999,973. |
-

122. In factoring $a^6 - b^6$, the expression may be considered either the difference of two squares or the difference of two cubes, producing respectively the following results:

$$a^6 - b^6 = (a^3 + b^3)(a^3 - b^3)$$

$$a^6 - b^6 = (a^2 - b^2)(a^4 + a^2 b^2 + b^4).$$

The factors of the first result can easily be factored again, while it is difficult to factor the last factor of the second result. Hence the prime factors of examples of this type can be obtained most readily by considering them the difference of two squares.

$$\begin{aligned} \text{Ex. } a^6 - 64 &= (a^3 + 8)(a^3 - 8) \\ &= (a + 2)(a^2 - 2a + 4)(a - 2)(a^2 + 2a + 4). \end{aligned}$$

EXERCISE 48

MISCELLANEOUS EXAMPLES

Resolve into prime factors:

- | | | |
|------------------|----------------------|------------------------|
| 1. $1 - x^6$. | 3. $a^6 b^6 - 729$. | 5. $1 - m^6 c^{12}$. |
| 2. $m^6 - n^6$. | 4. $a^{12} - b^6$. | 6. $a^{12} - b^{12}$. |

- | | |
|--------------------------------|----------------------------------|
| 7. $x^4 - 4x^2 + 3.$ | 17. $a^6b^2 - 26a^3b^2 - 27b^2.$ |
| 8. $x^4 - 2x^2 - 3.$ | 18. $x^6 - 5x^3 + 4.$ |
| 9. $3a^4 - 3a^2 - 36.$ | 19. $x^{17} - x.$ |
| 10. $4m^6 - 32m^4 - 36m^2.$ | 20. $a^{12} - 5a^6 + 4.$ |
| 11. $a^4 - 5a^2b^2 - 36b^4.$ | 21. $729a^8b^2 - a^2b^8.$ |
| 12. $a^5 + 4a^3 - 5a.$ | 22. $(a + b)^2 - 1.$ |
| 13. $a^7 - ab^6.$ | 23. $(a + b)^3 - 1.$ |
| 14. $3p^6 - 39p^4 + 108p^2.$ | 24. $(x + y)^4 - 1.$ |
| 15. $18x^4 - 74x^2y^2 + 8y^4.$ | 25. $3^{6n} - 2^{6n}.$ |
| 16. $x^6 - 7x^3 - 8.$ | |

TYPE VII. GROUPING TERMS

123. By the introduction of parentheses, polynomials can frequently be transformed into bi- and trinomials, which may be factored according to types I-VI.

A. After grouping the terms, we find that the new terms contain a common factor.

Ex. 1. Factor $ax + bx + ay + by.$

$$\begin{aligned} ax + bx + ay + by &= x(a + b) + y(a + b) \\ &= (a + b)(x + y). \end{aligned}$$

Ex. 2. Factor $x^3 - 5x^2 - x + 5.$

$$\begin{aligned} x^3 - 5x^2 - x + 5 &= x^2(x - 5) - (x - 5) \\ &= (x - 5)(x^2 - 1) \\ &= (x - 5)(x + 1)(x - 1). \end{aligned}$$

EXERCISE 49

- | | |
|-----------------------------|-----------------------------|
| 1. $xm + ym + xn + yn.$ | 4. $2am + 2ap - 3bm - 3bp.$ |
| 2. $x^2 + xy + ax + ay.$ | 5. $6am - 3bm - 6an + 3bn.$ |
| 3. $2ax - 3bx + 2ay - 3by.$ | 6. $2y^4 - y^3 + 4y - 2.$ |

- | | |
|--|------------------------------------|
| 7. $p^3q^3 - p^2q^2 - pq + 1.$ | 14. $a^3 - a - a^2xy + xy.$ |
| 8. $x^2 + mxy - 4xy - 4my^2.$ | 15. $p^3 + pq^2 - p^2q - q^3.$ |
| 9. $6x^2 + 3xy - 2ax - ay.$ | 16. $1 - x - x^2 + x^3.$ |
| 10. $c^2d^2 + e^2d^2 - c^2f^2 - e^2f^2.$ | 17. $p^3 - 5p^2 + 2p - 10.$ |
| 11. $3x^3 - 7x^2 + 3x - 7.$ | 18. $a^6 - a^5x - ap + px.$ |
| 12. $6x^4 - 13x^3 - 12x + 26.$ | 19. $m^6 - 13m^4 - 7m^2 + 91.$ |
| 13. $x^3 - x^2 - x + 1.$ | 20. $ax + bx + ay + by + az + bz.$ |

It is sometimes necessary to change the order of the terms of the given expression before the method can be applied.

$$21. \quad ax + by - ay - bx.$$

$$22. \quad x^3 - 21 + 3x - 7x^2.$$

$$23. \quad a^2x^3 - 3b^2y^2 + 3b^2x^3 - a^2y^2.$$

B. By grouping, the expression becomes the difference of two squares.

Ex. 1. Factor $a^2 - 6ab + b^2 - 16c^2$.

$$\begin{aligned} a^2 - 6ab + 9b^2 - 16c^2 &= (a^2 - 6ab + 9b^2) - 16c^2 \\ &= (a - 3b)^2 - (4c)^2 \\ &= (a - 3b + 4c)(a - 3b - 4c). \end{aligned}$$

Ex. 2. Factor $9x^2 - y^2 - 4z^2 + 4yz$.

$$\begin{aligned} 9x^2 - y^2 - 4z^2 + 4yz &= 9x^2 - (y^2 - 4yz + 4z^2) \\ &= (3x)^2 - (y - 2z)^2 \\ &= (3x + y - 2z)(3x - y + 2z). \end{aligned}$$

Ex. 3. Factor $4a^2 - b^2 + 9x^2 - 4y^2 - 12ax + 4by$.

Arranging the terms,

$$\begin{aligned} 4a^2 - b^2 + 9x^2 - 4y^2 - 12ax + 4by & \\ &= 4a^2 - 12ax + 9x^2 - b^2 + 4by - 4y^2 \\ &= (4a^2 - 12ax + 9x^2) - (b^2 - 4by + 4y^2) \\ &= (2a - 3x)^2 - (b - 2y)^2 \\ &= (2a - 3x + b - 2y)(2a - 3x - b + 2y). \end{aligned}$$

EXERCISE 50

1. $a^2 - 2ab + b^2 - 1$.
2. $a^2 - 4ab + 4b^2 - c^2$.
3. $x^2 - 6xy + 9y^2 - 25a^2$.
4. $1 - a^2 + 2ab - b^2$.
5. $16m^2 - 4x^2 - 4xy - y^2$.
6. $16 - a^2 - b^2 + 2ab$.
7. $25c^2 - 2xy - x^2 - y^2$.
8. $a^2 + b^2 + 2ab - x^2 + 2xy - y^2$.
9. $16x^4 - 4a^2 - 4ab - b^2$.
10. $3 - 3a^2 + 6ab - 3b^2$.
11. $a - a^3 - 6a^2b - 9ab^2$.
12. $x^2 - 10x + 25 - 121c^6$.
13. $a^2 + 12a + 36 - x^2 + 4xy - 4y^2$.
14. $a^2 - x^2 + b^2 - y^2 - 2ab - 2xy$.
15. $a^4 - 2a^2b^2 + b^4 - 25x^4 - 40x^2y^2 - 16y^4$.

C. By grouping, the expression becomes a trinomial of the form of $px^2 + qx + r$ (or its special cases II and IV).

Ex. 1. Factor $3x^2 - 6xy + 3y^2 - 10x + 10y + 3$.

$$\begin{aligned}
 3x^2 - 6xy + 3y^2 - 10x + 10y + 3 &= 3(x^2 - 2xy + y^2) - 10(x - y) + 3 \\
 &= 3(x - y)^2 - 10(x - y) + 3 \\
 &= [3(x - y) - 1][(x - y) - 3] \\
 &= [3x - 3y - 1][x - y - 3].
 \end{aligned}$$

Similar, although a type of VII A is the following example:

Ex. 2. $4x^2 - 12xy + 9y^2 - 2x + 3y$.

$$\begin{aligned}
 4x^2 - 12xy + 9y^2 - 2x + 3y &= (2x - 3y)^2 - (2x - 3y) \\
 &= (2x - 3y)(2x - 3y - 1).
 \end{aligned}$$

EXERCISE 51

1. $a^2 + 2a + 1 + ab + b$.
2. $a^2 - 2ab + b^2 - a + b$.
3. $a^2 - 4ab + 4b^2 - 3a + 6b + 2$.
4. $x^2 - y^2 + 2x - 2y$.
5. $x^2 - 3xy + 2y^2 - 2x + 4y$.
6. $x^2 - 6xy + 9y^2 - 7x + 21y + 12$.
7. $4a^2 + 8ab + 4b^2 - 5a - 5b + 1$.

(For additional methods of factoring, see Appendix II and Chapter XVI.)

SUMMARY OF FACTORING

I. First find monomial factors common to all terms.

II. Binomials are factored by means of the formulæ

$$a^2 - b^2 = (a + b)(a - b).$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

III. Trinomials are factored by the method of cross products, although frequently the particular cases II and IV are more convenient.

IV. Polynomials are reduced to the preceding cases by grouping terms.

EXERCISE 52

MISCELLANEOUS EXAMPLES

Factor the following expressions:

- | | |
|-------------------------------|--------------------------------|
| 1. $a^2 - b^2$. | 14. $17a^2 - 25a - 18$. |
| 2. $a^2 + b^2 - 2ab$. | 15. $20a^2 - 220a + 605$. |
| 3. $a^2 + 2ab$. | 16. $a^7 - 729a$. |
| 4. $a^2 - 3ab + 2b^2$. | 17. $7a^3 - 7$. |
| 5. $3a^2 - 10ab + 3b^2$. | 18. $7a^3 + 77a^2b - 84ab^2$. |
| 6. $6ab - 3a^2 - 3b^2$. | 19. $a^2 - c^2 + b^2 - 2ab$. |
| 7. $a^3 - a^2b$. | 20. $2a^2 + 16ab - 130b^2$. |
| 8. $a^3 - ab^2$. | 21. $a^2 - 1 - 2b - b^2$. |
| 9. $2xy - x^2 - y^2$. | 22. $x^2 - ax - bx + ab$. |
| 10. $x^3 - 16x^2 + 55x$. | 23. $2x^3 + x^4 - 3x^2$. |
| 11. $a^5 - 16a$. | 24. $2 - x - x^2$. |
| 12. $8x^4 - 65x^3 + 8x^2$. | 25. $(a + b + c)^2 - 1$. |
| 13. $3ax + 2bx + 3ay + 2by$. | 26. $1 - (a + b)^3$. |

- | | |
|--|---|
| 27. $x^9 - y^9$. | 42. $x^3a^2 - b^3a^2 - x^3y^2 + b^3y^2$. |
| 28. $7ab^3 - 7ab$. | 43. $x^3 - 3x^2 - x + 3$. |
| 29. $8x^4 - 31x^2 - 4$. | 44. $x^2 - 2x + 1 - y^2$. |
| 30. $a^2 - 1 + 2ab + b^2$. | 45. $x^4 - 5x^3 + x - 5$. |
| 31. $9x^2 - 9x^5$. | 46. $a^2 - b^2 + a - b$. |
| 32. $x^2y^2 + x^2 - 7y^2 - 7$. | 47. $x^2 - 172x + 171$. |
| 33. $(a+b)^2 - 9(a+b) + 20$. | 48. $3x^4 - 3x^3 - 720x^2$. |
| 34. $3ax^2 - 3ay^2 + 5bx^2 - 5by^2$. | 49. $225x^4 - 120x^2y^2 + 16y^4$. |
| 35. $27x^4y + xy$. | 50. $a^3 + a^2b + ab^2 + b^3$. |
| 36. $9x^2 - 82x + 9$. | 51. $x^2 + (a-b)x - ab$. |
| 37. $(a+b)^3 - 8$. | 52. $x^3 + 4x^2 - x - 4$. |
| 38. $729x^{30} - 100y^{60}$. | 53. $x^2 + 2x + 1 - 4(x+1)$. |
| 39. $289x^3 - 34x^2y + xy^2$. | 54. $a^9 - a$. |
| 40. $128 - 2a^2$. | 55. $x^5 - 8x^3 + 16x$. |
| 41. $10a^4 - 33a^3 - 7a^2$. | 56. $36a^{2m} - 9b^{2n}$. |
| 57. $x^2(a+b) + 4x(a+b) + 4(a+b)$. | |
| 58. $a^2 - 2ab + b^2 - 6a + 6b$. | |
| 59. $a^2 - 4ab + 3b^2 - 6a + 6b$. | |
| 60. $a^2 - 2ab + b^2 - 6a + 6b + 5$. | |
| 61. $6x^2 - 12xy + 6y^2 - 37x + 37y + 6$. | |
| 62. $x^3 - y^3 + x^2 - y^2$. | |
| 63. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$. | |

Simplify the following expressions, and factor the result:

64. $(x-1)(x-2) - 6$.
65. $(x+4)(x-3) - 18$.
66. $(x+7)(x-2) - (2x-9)(3x+2) + (x-3)(x-10) - 30$.

CHAPTER VII

HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

HIGHEST COMMON FACTOR

124. The highest common factor (H. C. F.) of two or more expressions is the algebraic factor of highest degree common to these expressions; thus a^6 is the H. C. F. of a^7 and a^6b^7 .

Two expressions which have no common factor except unity are *prime to one another*.

125. The H. C. F. of two or more monomials whose factors are prime can be found by inspection.

The H. C. F. of a^4 and a^2b is a^2 .

The H. C. F. of $a^2b^3c^4$, $a^3b^2c^4$, and a^4b^3 is a^2b^2 .

The H. C. F. of $(a+b)^3$ and $(a+b)^2(a-b)^4$ is $(a+b)^2$.

126. If the expressions have numerical coefficients, find by arithmetic the greatest common factor of the coefficients, and prefix it as a coefficient to H. C. F. of the algebraic expressions. Thus the H. C. F. of $6x^4yz$, $12x^3y^3z$, and $60x^3y^3$ is $6x^3y$.

The student should note that the power of each factor in the H. C. F. is the lowest power in which that factor occurs in any of the given expressions.

EXERCISE 53

Find the H. C. F. of:

- | | |
|-----------------------------------|--|
| 1. $6a^3b$, $2a^2b^3$. | 5. $4a^2x^2$, $6a^3x^3$, $12ax^2$. |
| 2. $9a^2b^4c^5$, $15a^3b^2c^5$. | 6. $19x^2y^3$, $95xz^3$, $117xy^4$. |
| 3. $17a^4b^3c$, $51c^4d$. | 7. $12x^3y^4z^5$, $18x^4y^5z^6$, $24x^2y^2z^6$. |
| 4. $24x^2yz$, $6y^2z^3u$. | 8. $3x^2y^3z$, $9x^4y^6z$, $15x^2y^5z$. |

9. $4 a^2 b^3 c^4$, $16 a^4 b^3 d^4$, $64 a^3 b^3 e^4$.
 10. $98 a^2 b^3 c^4$, $180 a^3 b^4 c^6$, $-300 a^4 b^3 c^5$.
 11. $15 a^2 b^2 x^2 y^2$, $-45 b^3 y^4$, $-90 a^4 b^4 x^4 y^4$.
 12. $3(a+b)^3$, $4(a+b)^2$, $3(a+b)^2(a-b)$.
 13. $3(x+1)(x+2)$, $12(x+1)(x+3)$, $6(x+1)^2$.
 14. $(a+b)^9(c+d)^8$, $(a+b)(c+d)^2$, $(a+b)^2(c+d)^3$.
 15. $6(x+y)^3$, $8(x+y)^2(x-y)$, $9(x+y)(x-y)^2$.
 16. $6 a^2(a+b)^9$, $8 a(a+b)^8$, $10 a^3(a+b)^7$.
-

127. To find the H.C.F. of polynomials, resolve each polynomial into prime factors, and apply the method of the preceding article.

Ex. 1. Find the H.C.F. of $x^2 - 4xy + 4y^2$, $x^2 - 3xy + 2y^2$, and $x^2 - 7xy + 10y^2$.

$$x^2 - 4xy + 4y^2 = (x - 2y)^2.$$

$$x^2 - 3xy + 2y^2 = (x - 2y)(x - y).$$

$$x^2 - 7xy + 10y^2 = (x - 2y)(x - 5y).$$

Hence the H.C.F. $\quad \quad \quad = x - 2y$.

Ex. 2. Find the H.C.F. of $6a^4 - 6ab^3$, $2a^3 - 8a^2b + 6ab^2$, and $12a^4 - 12a^2b^2$.

$$6a^4 - 6ab^3 = 6a(a^3 - b^3) = 6a(a - b)(a^2 + ab + b^2).$$

$$2a^3 - 8a^2b + 6ab^2 = 2a(a^2 - 4ab + 3b^2) = 2a(a - b)(a - 3b).$$

$$12a^4 - 12a^2b^2 = 12a^2(a^2 - b^2) = 12a^2(a + b)(a - b).$$

Hence the H.C.F. $\quad \quad \quad = 2a(a - b)$.

128. If $7a$ is contained in several expressions, obviously $-7a$ must be contained also. Similarly, if $a - b$ is a common factor of several expressions, $-a + b$, or $b - a$, is a common factor also. From this it follows that each set of expressions has really two highest common factors, whose absolute values are equal, but whose signs differ. *E.g.* Ex. 1 of § 127 has the two answers, $x - 2y$ and $2y - x$; Ex. 2, $2a(a - b)$ and $-2a(a - b)$ or $2a(b - a)$.

EXERCISE 54

Find the H. C. F. of:

1. $3x^3y^4, 12x^2y^3 - 18x^3y^2.$
2. $12x^2y^5z^5, 3x^4y^5z^6 - 18x^3y^6z^5.$
3. $36a^2x, 12a^3x^2 + 24a^2x^3.$
4. $9ax - 12a^2x^2, 24a^2x^2 + 36a^3x^3.$
5. $a^2 - b^2, 4a^2b + 4ab^2.$
6. $4a^2 - 9b^2, 10a^3b + 15a^2b^2.$
7. $9ax^2 - 16ay^2, 12abx + 16aby.$
8. $16a^2b^2 - 25b^2c^2, 24ab^2 - 30b^2c.$
9. $4x^2 + 12xy + 9y^2, 16x + 24y.$
10. $m^2 - 4x^2, m^2 + 2mx.$
11. $m^2 - n^2, m^2 + mn, m^2n + mn^2.$
12. $4x^3 + 12x^2y + 9xy^2, 16xy + 24y^2.$
13. $9a^3 + 24a^2b + 16ab^2, 18a^3 + 24a^2b.$
14. $4a^2x^2 + 12a^2xy + 9a^2y^2, 18a^3x + 27a^3y.$
15. $a + b, a^4 - b^4, a^3 + b^3.$
16. $ax + ay - bx - by$ and $a^4 - b^4.$
17. $x^2 - 7x + 12, x^2 - 8x + 15.$
18. $x^2 - 3x - 4, x^2 - 8x + 16, x^3 - 16x.$
19. $a^2 + 3x - 18, a^3 - 27, a^2 - 6a + 9.$
20. $a^2 + 2a - 3, a^2 + 7a + 12, a^4 + 27a.$
21. $x^2 + 3x - 54, x^2 + x - 42, x^2 + 2x - 48.$
22. $2a^2 + 9a + 4, 2a^2 + 11a + 5, 2a^2 - 3a - 2.$
23. $3a^3 + 15a^2b + 18ab^2, 3a^4 + 9a^3b + 6a^2b^2.$
24. $a^2 + 4ab + 3b^2, a^2 + 2ab - 3b^2, a^2 + 9ab + 18b^2.$

25. $x^2 - 4xy + 4y^2$, $x^3 - 8y^3$, $x^4 - 16y^4$.
 26. $x^3 - 2x^2 - 3x + 6$, $2x^2 - 5x + 2$.
 27. $3a^3b - 3a^2b - 21ab$, $7a^2 - 7a - 49$.
 28. $a - b$, $-a + b$.
 29. $a - 3x$, $3x - a$.
 30. $13a^2 - 13b^2$, $26b^3 - 26a^3$.
 31. $3x^2 - 10x + 3$, $9x - x^3$.

129. If only one of the given expressions can be factored by inspection, determine by actual division if its factors are contained in the remaining expressions.

Ex. 1. Find the H. C. F. of $x^2 - 4$, and $x^3 - 3x^2 + 8x - 12$.

The factors of $x^2 - 4$ are $x - 2$, and $x + 2$. By dividing $x^3 - 3x^2 + 8x - 12$ we find that $x - 2$ is a factor, but $x + 2$ is not a factor. Hence the H. C. F. = $x - 2$.

130. If several but not all expressions can be factored by inspection, find the H. C. F. of those which can be factored, and test the factors of the H. C. F. by actual division as in the preceding case.

Ex. 2. Find the H. C. F. of $x^3 - 3x^2 - 8x + 24$, $x^3 - 27$, and $x^3 - 2x^2 + 6x - 27$.

$$x^3 - 3x^2 - 8x + 24 = x^2(x - 3) - 8(x - 3) = (x - 3)(x^2 - 8).$$

$$x^3 - 27 = (x - 3)(x^2 + 3x + 9).$$

The H. C. F. of these two expressions is $x - 3$.

$$\begin{array}{r|l} x^3 - 2x^2 + 6x - 27 & x - 3 \\ \hline x^3 - 3x^2 & x^2 + x + 9 \\ \hline x^2 + 6x & \\ x^2 - 3x & \\ \hline 9x - 27 & \\ 9x - 27 & \\ \hline 0 & \end{array}$$

Hence the H. C. F. = $x - 3$.

131. If divisor and dividend are both arranged, the division can be exact only when the first term and last term of the divisor are respectively divisible by the first and last terms of the dividend.

Ex. 3. Find the H.C.F. of $x^3 + 3x^2 - 11x - 26$ and $(3x - 1)(x - 3)(x + 2)$.

Since x^3 is not exactly divisible by $3x$, $3x - 1$ has to be rejected, and since 26 is not exactly divisible by -3 , $x - 3$ has to be rejected. Hence only $x + 2$ has to be tried. But actual division shows that $x + 2$ is a factor of $x^3 + 3x^2 - 11x - 26$. Therefore $x + 2$ is the H.C.F.

EXERCISE 55

Find the H.C.F. of:

1. $a^2 - 3a + 2$, $a^4 - 6a^2 + 8a - 3$.
2. $4a^2 - 1$, $4a^3 - 6a^2 - 4a + 3$.
3. $a^2 - 3a + 2$, $a^2 + a - 2$, $a^3 + 2a^2 - a - 2$.
4. $2x^4 + 5x^3 + 2x^2$, $x^4 - 9x^3 + 9x - 70$.
5. $x^2 - 4x + 3$, $3x^2 - 10x + 3$, $x^3 - 6x^2 + 11x - 6$.
6. $8x^3 - 1$, $4x^2 - 1$, $4x^3 + x - 1$.
7. $8x^3 - 1$, $8x^2 + 4x + 2$, $16x^4 + 4x^2 + 1$.
8. $x^3 - x$, $x^4 - 7x^2 + 6$, $x^4 - 3x^3 + 5x^2 + 3x - 6$.
9. $x^2 + 5x - 24$, $x^3 + 4x^2 - 26x + 15$.
10. $3a^2 + a - 2$, $4a^3 + 4a^2 - a - 1$.

(For another method of finding the H.C.F. of expressions which cannot be factored by inspection, see Appendix III.)

LOWEST COMMON MULTIPLE

132. A common multiple of two or more expressions is an expression which can be divided by each of them without a remainder.

Common multiples of $3x^2$ and $5y$ are $30x^2y$, $60x^2y^2$, $300x^2y$, etc.

133. The lowest common multiple (L.C.M.) of two or more expressions is the common multiple of lowest degree; thus, x^3y^3 is the L.C.M. of x^3y and xy^3 .

134. The L.C.M. of two or more monomials whose factors are prime can be found by inspection.

The L. C. M. of a^4b^6 and a^5b is a^5b^6 .

135. If the expressions have a numerical coefficient, find by arithmetic their least common multiple and prefix it as a coefficient to the L.C.M. of the algebraic expressions.

The L. C. M. of $3a^3b^3$, $2a^2b^2c^3$, $6c^6$ is $6a^3b^3c^6$.

The L. C. M. of $12(a+b)^3$ and $(a+b)^2(a-b)^2$ is $12(a+b)^3(a-b)^2$.

136. Obviously the power of each factor in the L.C.M. is equal to the highest power in which it occurs in any of the given expressions.

137. To find the L.C.M. of several expressions which are not completely factored, resolve each expression into prime factors and apply the method for monomials.

Ex. 1. Find the L.C.M. of $4a^2b^2$ and $4a^4 - 4a^2b^3$.

$$4a^2b^2 = 4a^2b^2.$$

$$4a^4 - 4a^2b^3 = 4a^2(a^2 - b^3).$$

$$\text{Hence, L. C. M.} \quad = 4a^2b^2(a^2 - b^3).$$

Ex. 2. Find the L. C. M. of $a^2 - b^2$, $a^2 + 2ab + b^2$, and $b - a$.

$$a^2 - b^2 = (a + b)(a - b).$$

$$a^2 + 2ab + b^2 = (a + b)^2.$$

$$b - a = -(a - b).$$

$$\text{Hence the L. C. M.} \quad = (a + b)^2(a - b).$$

NOTE. The L. C. M. of the last example is also $-(a + b)^2(a - b)$. In general, each set of expressions has two lowest common multiples, which have the same absolute value, but opposite signs.

EXERCISE 56

Find the L. C. M. of:

1. x, x^2, x^3 .
2. $x, 3xy, 5xy^2$.
3. $x^2, 3x^2y, 5xz^2, 12$.
4. $2a^2x, 3b^2y, 4a^3x^2$.
5. $12, 15a^2, 25ab^2, 10b^3, 18a^2b$.
6. $8a^2bx, 12a^5x^3, 14y^3$.
7. $5x^3y^2, 4y^2, 20x^2y^3, 45x^3$.
8. $13x^2y^2, 15x^3y^3, 9xy^4, c^4$.
9. $4a^5bcd, 20ab^5cd^2, 40abc^5d, 8d^5$.
10. $2a^2, 4a^4, 6a^6, b$.
11. $9x, 3xy, 2(x+y)$.
12. $4(x-y)^2, 3(x+y)^3, 12(x-y)(x+y)$.
13. $(a+b)^m, (a+b)^{m+1}, (a+b)^{m+2}$.
14. $(x-1)(x-2)^2, (x-2)(x-3)^2, (x-3)(x-1)^2$.
15. $3x, 5y, x^2y - xy^2$.
16. $4x^2, 3y^2, 6x^3y^2 - 6x^2y^3$.
17. $6(x+y), 3x-3y, 6xy+6y^2$.
18. $4abc, 9a^2-16x^2$.
19. a^2-b^2, a^3-b^3 .
20. $a^3+b^3, a^2-ab+b^2, a+b$.
21. $x^2+5x+6, x^2+4x+3, x^2+x$.
22. $4x^4-4x^2, 3x^2+6x+3, 6x^2-12x+6$.
23. $4a^2+2a-2, 4a^2-1, 4a^2+4a-3$.
24. $3x^2+3x, 5x^2-5, 15x$.
25. $x+1, x+2, x+3$.
26. $2x-1, 4x+2, 4x^2-1$.
27. $2x^2-x-1, 2x^2+x-3, 4x^2+8x+3$.
28. $1-x, x-1, x^2-1, x^4-1, x^8-1$.

29. $6x^2 - 54, 7(x - 3)^2, 3x^2 - 9x + 3.$

30. $x^3 - x^2 + x - 1, x^3 - x.$

31. $3a^2 - 11a + 6, 3a^2 - 8a + 4, a^2 - 5a + 6.$

32. $2x^3 - 3x^2 - 2x, 2x^3 - 15x^2 - 8x, x^2 - 10x + 16.$

33. $12a^2 + 23ab + 10b^2, 4a^2 + 9ab + 5b^2, 3a^2 + 5ab + 2b^2.$

34. $(a - b)(b - c), (b - c)(c - a), (c - a)(b - a).$

35. $(a + b)^2 - c^2, a^2 - (b + c)^2.$

36. $x^2 - 1, x^3 - 2x^2 + 4x - 3.$

(The method for finding the L. C. M. of expressions which cannot be factored by inspection will be found in Appendix III.)

CHAPTER VIII

FRACTIONS

REDUCTION OF FRACTIONS

138. A fraction is an indicated quotient; thus $\frac{a}{b}$ is identical with $a \div b$. The dividend a is called the *numerator* and the divisor b the *denominator*. The numerator and the denominator are the *terms* of the fraction.

139. Since a fraction represents an indicated division, the proofs of all fundamental properties may be based upon the definition of division; viz.:

$$\frac{a}{b} = x \text{ means } a = bx.$$

140. If both of the terms of a fraction be multiplied by the same expression, the value of the fraction is not thereby altered.

Expressed in symbols,

$$\text{if } \frac{a}{b} = x, \text{ then } \frac{ma}{mb} = x.$$

According to the preceding paragraph, this really means
if $a = bx$, then $ma = mbx$.

But in this form the correctness of the conclusion is obvious.
(§ 90, 3).

Hence it is proved that $\frac{a}{b} = \frac{ma}{mb}$.

141. If both the terms of a fraction be divided by the same expression, the value of the fraction is not thereby altered; or

$$\frac{ma}{mb} = \frac{a}{b}.$$

This follows from the preceding article.

142. A fraction is in its **lowest terms** when its numerator and its denominator have no common factors.

Ex. 1. Reduce $\frac{6xy^2z^4}{9x^2y^2z^3}$ to its lowest terms.

Remove successively all common divisors of numerator and denominator, as 3, x , y^2 , and z^3 (or divide the terms by their H. C. F. $3xy^2z^3$).

Hence
$$\frac{6xy^2z^4}{9x^2y^2z^3} = \frac{2z}{3x}.$$

143. To reduce a fraction to its lowest terms, *resolve numerator and denominator into their factors, and cancel all factors that are common to both. Never cancel terms of the numerator or the denominator; cancel factors only.*

Ex. 2. Reduce $\frac{a^3 - 6a^2 + 8a}{6a^4 - 24a^2}$ to its lowest terms.

$$\begin{aligned} \frac{a^3 - 6a^2 + 8a}{6a^4 - 24a^2} &= \frac{a(a^2 - 6a + 8)}{6a^2(a^2 - 4)} \\ &= \frac{a(a-2)(a-4)}{6a^2(a+2)(a-2)} \\ &= \frac{a-4}{6a(a+2)}. \end{aligned}$$

Ex. 3. Reduce $\frac{-6a^4 + 60a^3b + 66a^2b^2}{-12a^{10}(a^2 - 121b^2)(a^2 - b^2)}$ to its lowest terms.

$$\begin{aligned} \frac{-6a^4 + 60a^3b + 66a^2b^2}{-12a^{10}(a^2 - 121b^2)(a^2 - b^2)} &= \frac{-6a^2(a^2 - 10ab - 11b^2)}{-12a^{10}(a+11b)(a-11b)(a+b)(a-b)} \\ &= \frac{-6a^2(a-11b)(a+b)}{-12a^{10}(a+11b)(a-11b)(a+b)(a-b)} \\ &= \frac{1}{2a^8(a+11b)(a-b)}. \end{aligned}$$

144. By the law of signs in division

$$\frac{+a}{+b} = \frac{-a}{-b} = -\frac{+a}{-b} = -\frac{-a}{+b}.$$

That is, a fraction is not altered if the signs of both its terms are changed; but the sign of the fraction is changed if the sign of either term is changed.

Ex. 4. Reduce $\frac{a^2 + ab - 2b^2}{b^2 - a^2}$ to its lowest terms.

$$\begin{aligned}\frac{a^2 + ab - 2b^2}{b^2 - a^2} &= \frac{a^2 + ab - 2b^2}{-(a^2 - b^2)} \\ &= -\frac{a^2 + ab - 2b^2}{a^2 - b^2} \\ &= -\frac{(a-b)(a+2b)}{(a+b)(a-b)} \\ &= -\frac{a+2b}{a+b}.\end{aligned}$$

EXERCISE 57

Reduce to lowest terms:

- | | | |
|---|---|---------------------------------------|
| 1. $\frac{9x^4}{3x^3}$ | 6. $\frac{25x^3}{125x^3y}$ | 11. $\frac{x^2y - xy^2}{9x - 9y}$ |
| 2. $\frac{48a^3b^5}{16a^2b^2}$ | 7. $\frac{17a^{n+2}}{16a^n}$ | 12. $\frac{6ab}{6a^2b - 6ab^2}$ |
| 3. $\frac{250x^3y^3z^7}{25x^5y^4z^7}$ | 8. $\frac{35a^{n+2}b^{m+3}}{7a^2b^2}$ | 13. $\frac{ax + bx}{cx}$ |
| 4. $\frac{34a^5b^5c^5}{85a^7b^9x^4}$ | 9. $\frac{12m + 12n}{24x + 24y}$ | 14. $\frac{25rx - 35ry}{35sx - 49sy}$ |
| 5. $\frac{23a^9x^7}{46a^4x^4y^2}$ | 10. $\frac{9a^2b + 12ab^2}{9a^3b - 15ab^3}$ | 15. $\frac{4a^2 - 9b^2}{2a + 3b}$ |
| 16. $\frac{4a^2 - 9b^2}{(2a - 3b)^2}$ | 21. $\frac{1 + a^3}{3a^2 - 3a^3 + 3a^4}$ | |
| 17. $\frac{ax + bx - cx}{ay + by - cy}$ | 22. $\frac{3x^2 - 10x + 3}{9x^2 - 6x + 1}$ | |
| 18. $\frac{m^2 - 2mn + n^2}{m^2 - n^2}$ | 23. $\frac{4a^2 + 12ab + 9b^2}{16ab + 24b^2}$ | |
| 19. $\frac{9x^2 + 6x + 1}{9x^2 - 1}$ | 24. $\frac{a^2 - 7a + 12}{a^2 - 8a + 15}$ | |
| 20. $\frac{a^3b^3c^3 + 1}{a^2b^2c^2 + abc}$ | 25. $\frac{a^2 - 3a - 4}{a^2 - 8a + 16}$ | |

$$26. \frac{a^2 - 7a + 12}{a^2 - 2a - 3}.$$

$$27. \frac{2a^2 + 9a + 4}{2a^2 + 11a + 5}.$$

$$28. \frac{15x^2 + 2xy - y^2}{12x^2 + xy - y^2}.$$

$$29. \frac{2m^2 - mn - 6n^2}{3m^2 - 7mn + 2n^2}.$$

$$30. \frac{ax + ay - cx - cy}{ax + ay + cx + cy}.$$

$$31. \frac{2ac - 2ad - 3bc + 3bd}{2ac - 2ad + 3bc - 3bd}.$$

$$32. \frac{mx + m - x - 1}{m^3 - 1}.$$

$$33. \frac{mn - 2m + 3n - 6}{mn - 2m - 3n + 6}.$$

$$42. \frac{x^2 - 4x + 4}{-x^2 + 5x - 6}.$$

$$34. \frac{2p^2 - pq - 3q^2}{2p^3 - 5p^2q + 3pq^2}.$$

$$35. \frac{a^2 + b^2 + 2ab - c^2}{a^2 + c^2 + 2ac - b^2}.$$

$$36. \frac{3x^4 - 21x^3 + 36x^2}{x^2 - 8x + 15}.$$

$$37. \frac{a^4 - b^4}{a^3 + b^3}.$$

$$38. \frac{a^4 - b^4}{a^6 - b^6}.$$

$$39. \frac{(a+b)^3}{a^3 + b^3}.$$

$$40. \frac{7pq^3 - 7pq}{14pq^4 - 14pq}.$$

$$41. \frac{a-b}{b-a}.$$

$$43. \frac{a^2 - 9ab + 14b^2}{2b^2 + ab - a^2}.$$

$$44. \frac{a^{3m} - 1}{1 - a^{2m}}.$$

145. If only one of the terms of the fraction can be factored by inspection, find, if possible, the H. C. F. according to § 129, and divide numerator and denominator by it.

Ex. 1. Reduce $\frac{3a^3 - 13a^2 + 23a - 21}{9a^2 - 36a + 35}$ to its lowest terms.

$$9a^2 - 36a + 35 = (3a - 5)(3a - 7).$$

But $3a - 5$ cannot be a factor of the numerator. (Art. 131.)

Hence only $3a - 7$ need to be tried.

Actual division shows that $3a^3 - 13a^2 + 23a - 21$ is exactly divisible by $3a - 7$, and that the quotient is $a^2 - 2a + 3$.

$$\begin{aligned} \text{Therefore } \frac{3a^3 - 13a^2 + 23a - 21}{9a^2 - 36a + 35} &= \frac{(3a - 7)(a^2 - 2a + 3)}{(3a - 5)(3a - 7)} \\ &= \frac{a^2 - 2a + 3}{3a - 5}. \end{aligned}$$

146. Since a factor of each term of a fraction is also a factor of their difference or sum, we can frequently find the H.C.F. of these terms without factoring them, and thereby reduce the fraction.

Ex. 2. Reduce $\frac{2x^3 - 4x^2 + 5x - 33}{2x^3 - 7x^2 + 9x - 18}$ to its lowest terms.

The difference between numerator and denominator $= 3x^2 - 4x - 15$
 $= (3x + 5)(x - 3).$

$3x + 5$ is not contained in either term. (Art. 131.)

Hence if there is a H.C.F., it is $x - 3$.

By trial $x - 3$ is found to be the H.C.F., and the fraction reduces by division to $\frac{2x^2 + 2x + 11}{2x^2 - x + 6}.$

EXERCISE 58

Reduce to lowest terms:

$$1. \frac{x^3 - 2x^2 + 7x - 30}{3x^2 - 10x + 3}.$$

$$6. \frac{a^3 + 5a^2b + 8ab^2 + 6b^3}{a^3 + 3a^2b + 4ab^2 + 2b^3}.$$

$$2. \frac{x^3 - 7x + 6}{2x^3 - 5x^2 + 2x}.$$

$$7. \frac{3a^3 + 7a^2 - 8a - 2}{3a^3 + 2a^2 - 10a + 5}.$$

$$3. \frac{x^3 - 5x^2 - 8x + 48}{x^3 - 9x^2 + 20x}.$$

$$8. \frac{x^3 - 4x^2 + x + 6}{x^3 - 3x^2 + 3x - 2}.$$

$$4. \frac{x^3 + 6x^2 + 7x + 10}{x^3 + 5x^2 - x - 5}.$$

$$9. \frac{5x^3 - 3x^2y + 2xy^2 - 4y^3}{5x^3 - 9x^2y + 7xy^2 - 3y^3}.$$

$$5. \frac{x^3 - 6x^2 + 11x - 6}{x^3 - 8}.$$

$$10. \frac{2x^4 + x^3 - 2x - 1}{3x^4 + x^3 - 3x - 1}.$$

147. Reduction of fractions to equal fractions of lowest common denominator. Since the terms of a fraction may be multiplied by any quantity without altering the value of the fraction, we may use the same process as in arithmetic for reducing fractions to the lowest common denominator.

Ex. 1. Reduce $\frac{7a}{6b^2x^2}$, $\frac{2b}{3a^3}$, and $\frac{5xy}{4ab^2}$ to their lowest common denominator.

The L. C. M. of $6b^2x^2$, $3a^3$, and $4ab^2$ is $12a^3b^2x^2$.

To reduce $\frac{7a}{6b^2x^2}$ to the denominator $12a^3b^2x^2$, numerator and denominator must be multiplied by $\frac{12a^3b^2x^2}{6b^2x^2}$ or $2a^3$.

Similarly, multiplying the terms of $\frac{2b}{3a^3}$ by $4b^2x^2$, and the terms of $\frac{5xy}{4ab^2}$ by $3a^2x^2$, we have $\frac{14a^4}{12a^3b^2x^2}$, $\frac{8b^3x^2}{12a^3b^2x^2}$, and $\frac{15a^2x^3y}{12a^3b^2x^2}$.

148. To reduce fractions to their lowest common denominator, take the L. C. M. of the denominators for the common denominator. Divide the L. C. M. by the denominator of each fraction, and multiply each quotient by the corresponding numerator.

Ex. Reduce $\frac{3x}{(x+3)(x-3)}$, $\frac{3x-2}{(x-3)(x-1)}$, and $\frac{5}{(x-1)(x+3)}$ to their lowest common denominator.

The L. C. D. = $(x+3)(x-3)(x-1)$.

Dividing this by each denominator, we have the quotients $(x-1)$, $(x+3)$, and $(x-3)$.

Multiplying these quotients by the corresponding numerators and writing the results over the common denominator, we have

$$\frac{3x^2 - 3x}{(x+3)(x-3)(x-1)}, \quad \frac{3x^2 + 7x - 6}{(x+3)(x-3)(x-1)}, \quad \text{and} \quad \frac{5x - 15}{(x+3)(x-3)(x-1)}.$$

NOTE. Since $a = \frac{a}{1}$, we may extend this method to integral expressions.

EXERCISE 59

Reduce the following to their lowest common denominator:

1. $\frac{2a^2}{3x^2}, \frac{5x^2}{4a^2}$

3. $5, \frac{x^2}{a^2y^2}$

5. $\frac{x^2}{a^2}, \frac{1}{a^3}, \frac{4}{3a^4}$

2. $\frac{12x^2}{5c^2b^2}, \frac{7}{3ab^2}$

4. $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$

6. $\frac{a^2b}{4c^2d}, \frac{3ab}{2c^4d^3}$

7. $3x, \frac{x^3}{6}, \frac{6}{x^3}.$

10. $\frac{3}{x^2 - ax}, \frac{4}{x^3 - a^2x}.$

8. $\frac{3cy^3}{7a^3b^5x^2}, \frac{5x^5y^2}{2a^4b^2c^4}, \frac{2x^8}{3abc}.$

11. $\frac{3a+2}{3a-9}, \frac{2a^2+1}{a^2-9}.$

9. $\frac{1}{a-b}, \frac{a}{a^2-ab}.$

12. $\frac{3}{x+a}, \frac{5}{x-a}, \frac{6}{x^2-a^2}.$

13. $\frac{3a}{a^2+2ab+b^2}, \frac{4a}{a^2-2ab+b^2}, \frac{6a}{a^2-b^2}.$

14. $5x^2, \frac{1}{x^2-2x-3}, \frac{5}{x^2-9}.$

15. $\frac{x+3}{x^2-3x+2}, \frac{x+2}{x^2-4x+3}, \frac{x+1}{x^2-5x+6}.$

16. $\frac{p}{p-q}, \frac{6p}{p^2+pq+q^2}, \frac{9p}{p^3-q^3}.$

17. $\frac{4p}{p+q}, \frac{9}{p^2-pq+q^2}, \frac{7q}{p^3+q^3}.$

18. $\frac{a^2+b^2}{2ab}, \frac{2b}{a^2-ab}, \frac{ab}{ab-b^2}.$

ADDITION AND SUBTRACTION OF FRACTIONS

149. Since $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$ (Art. 74), **fractions having a common denominator are added or subtracted by dividing the sum or the difference of the numerators by the common denominator.**

150. If the given fractions have different denominators, they must be reduced to equal fractions which have the lowest common denominator before they can be added (or subtracted).

Ex. 1. Simplify $\frac{2a+3b}{2(2a-3b)} + \frac{6a-b}{4(2a+3b)}$.

The L. C. D. is $4(2a-3b)(2a+3b)$.

Multiplying the terms of the first fraction by $2(2a+3b)$, the terms of the second by $(2a-3b)$ and adding, we obtain

$$\begin{aligned} \frac{2a+3b}{2(2a-3b)} + \frac{6a-b}{4(2a+3b)} &= \frac{2(2a+3b)(2a+3b) + (2a-3b)(6a-b)}{4(2a-3b)(2a+3b)} \\ &= \frac{8a^2 + 24ab + 18b^2 + 12a^2 - 20ab + 3b^2}{4(2a-3b)(2a+3b)} \\ &= \frac{20a^2 + 4ab + 21b^2}{4(2a-3b)(2a+3b)}. \end{aligned}$$

151. The results of addition and subtraction should be reduced to their lowest terms.

Ex. 2. Simplify $\frac{a+2b}{a^2-ab} + \frac{2a+b}{a^2-3ab+2b^2} - \frac{a-3b}{a^2-2ab}$.

$$a^2 - ab = a(a-b),$$

$$a^2 - 3ab + 2b^2 = (a-b)(a-2b).$$

$$a^2 - 2ab = a(a-2b).$$

Hence the L. C. D. $= a(a-b)(a-2b)$.

$$\begin{aligned} \frac{a+2b}{a^2-ab} + \frac{2a+b}{a^2-3ab+2b^2} - \frac{a-3b}{a^2-2ab} &= \frac{(a+2b)(a-2b) + a(2a+b) - (a-3b)(a-b)}{a(a-b)(a-2b)} \\ &= \frac{a^2 - 4b^2 + (2a^2 + ab) - (a^2 - 4ab + 3b^2)}{a(a-b)(a-2b)} \\ &= \frac{a^2 - 4b^2 + 2a^2 + ab - a^2 + 4ab - 3b^2}{a(a-b)(a-2b)} \\ &= \frac{2a^2 + 5ab - 7b^2}{a(a-b)(a-2b)} = \frac{(2a+7b)(a-b)}{a(a-b)(a-2b)} \\ &= \frac{2a+7b}{a(a-2b)}. \end{aligned}$$

NOTE. In simplifying a term preceded by the minus sign, *e.g.* $-(a-3b)(a-b)$, the student should remember that parentheses are understood about terms (§ 68); hence he should, in the beginning, write the product in a parenthesis, as $-(a^2 - 4ab + 3b^2)$.

EXERCISE 60

Simplify:

$$1. \frac{5a-7b}{4} - \frac{3a-5b}{6} + \frac{11a-3b}{12}.$$

$$2. \frac{3a+4b}{9} - \frac{5a-7b}{15} - \frac{6a+11b}{45}.$$

$$3. \frac{a}{b} + \frac{c}{d}.$$

$$13. \frac{a+b}{6a} - \frac{a-b}{5b} + \frac{a-2b}{2a}.$$

$$4. \frac{1}{x} - \frac{1}{y}.$$

$$14. \frac{x-2y}{2x^2} - \frac{x-y}{2xy} + \frac{x+y}{3y^2}.$$

$$5. \frac{5bc}{3a} + \frac{3ac}{4b} - \frac{7ab}{2c}.$$

$$15. \frac{1}{x+3} + \frac{1}{x+4}.$$

$$6. \frac{2a+3b}{4a} - \frac{3a-7b}{5b}.$$

$$16. \frac{1}{x-3} - \frac{1}{x-4}.$$

$$7. \frac{6x-11y}{13x} - \frac{15y-8x}{17y}.$$

$$17. \frac{5}{x-1} + \frac{6}{x+1}.$$

$$8. \frac{7x+5y}{5x^2} - \frac{2y-3z}{7yz} + \frac{4y}{3xz}.$$

$$18. \frac{x}{x+y} - \frac{y}{x-y}.$$

$$9. \frac{2}{5a} + \frac{9a-2b}{15ab} - \frac{a+b}{10a^2}.$$

$$19. \frac{4a}{a-2b} - \frac{3b}{a-3b}.$$

$$10. \frac{7u-3v}{12uv} - \frac{5u+6v}{18v} - \frac{5uv-8v}{30u}.$$

$$20. \frac{1+x}{1-x} - \frac{1-x}{1+x}.$$

$$11. x + \frac{1}{y} \quad \left(\text{HINT: Let } x = \frac{x}{1} \right)$$

$$21. \frac{x-2y}{x+3y} + \frac{x-4y}{x-3y}.$$

$$12. \frac{2a+3}{6a^2} + \frac{a}{2b^2} + \frac{3a-2b}{6ab}.$$

$$22. \frac{x+y}{x-y} - \frac{x-3y}{x+y} + \frac{x^2-4y^2}{x^2-y^2}.$$

$$23. \frac{2a-3b}{2a+3b} - \frac{2a+3b}{2a-3b} + \frac{8a^2+18b^2}{4a^2-9b^2}.$$

$$24. \frac{4x^2+y^2}{4x^2-y^2} - \frac{4x-y}{4x+y}.$$

$$25. \frac{5x}{x-9} - \frac{6x}{x-3} + \frac{x^2-2x+1}{x^2-12x+27}.$$

$$26. \frac{a^2}{(a+b)^2} + \frac{a}{(a+b)} + 2.$$

$$27. \frac{a}{a^2+ab} + \frac{a+b}{ab-b^2} - \frac{b}{a^2-b^2}.$$

$$28. \frac{1}{a} - \frac{a}{2a-4} + \frac{a+2}{3a-6}.$$

$$29. \frac{2}{x-4} - \frac{3}{x-3} + \frac{1}{x}.$$

$$30. \frac{1}{a-1} - \frac{a}{a^2+a+1} - \frac{a^2+2}{a^3-1}.$$

$$31. a-1 + \frac{2}{a+1}.$$

$$32. a+b - \frac{a^2}{a-b}.$$

$$33. \frac{x-4}{2x-1} - \frac{3x-5}{x+2} + \frac{5x^2+9x+14}{2x^2+3x-2}.$$

$$34. \frac{4}{x-1} - \frac{3}{x-2} - \frac{1}{x-3}.$$

$$36. \frac{2}{x+m} + \frac{2}{x+b} - \frac{4}{x+c}.$$

$$35. \frac{6}{x-a} - \frac{4}{x-b} - \frac{2}{x-c}.$$

$$37. \frac{2}{(x-1)^3} + \frac{1}{(x-1)^2} + \frac{2}{x-1}.$$

$$38. \frac{16}{9(x-7)} - \frac{1}{4(x-2)} - \frac{19}{36(x+2)}.$$

$$39. \frac{a}{a^2-2ab+b^2} + \frac{b}{a^2+2ab+b^2} - \frac{2b-a}{a^2-b^2}.$$

$$40. \frac{x^3+x^2y}{x^2y-y^3} - \frac{x(x-y)}{xy+x^2} - \frac{2xy}{x^2-y^2}.$$

$$41. \frac{1}{x^2-3x+2} + \frac{1}{x^2-4x+3} + \frac{1}{x^2-5x+6}.$$

$$42. \frac{a^2 + ab + b^2}{a + b} + \frac{a^2 - ab + b^2}{a - b}.$$

$$43. \frac{3m}{1 - m^2} - \frac{3m}{1 + m^2}.$$

$$44. \frac{y}{x^3 + xy^2} + \frac{x}{x^2y + y^3} - \frac{2xy}{xy(x + y)}.$$

$$45. \frac{2}{1 - 3x} - \frac{3}{1 - 2x} + \frac{3}{x} - \frac{x}{1 - 5x + 6x^2}.$$

$$46. \frac{a^3}{a^2 - ab} - \frac{b^3}{a^2 + ab} - (a + b).$$

$$47. \frac{4a^2}{10ab - 25b^2} + \frac{25b^2}{4a^2 - 10ab} + \frac{2a}{5b} - \frac{5b}{2a}.$$

$$48. \frac{2x - 1}{4x^2 + 4x + 1} - \frac{2x + 1}{4x^2 - 4x + 1} - \frac{9(x^4 + 2)}{16x^4 - 8x^2 + 1}.$$

$$49. \frac{1}{a - 2} - \frac{1}{a + 2} + \frac{4}{a^2 + 4} - \frac{8a^2}{a^4 - 1}.$$

SOLUTION

Combining the first two fractions,

$$\frac{1}{a - 2} - \frac{1}{a + 2} = \frac{4}{a^2 - 4}. \quad (1)$$

Combining (1) with the third fraction,

$$\frac{4}{a^2 - 4} + \frac{4}{a^2 + 4} = \frac{8a^2}{a^4 - 16}. \quad (2)$$

Combining (2) with the last fraction,

$$\frac{8a^2}{a^4 - 16} - \frac{8a^2}{a^4 - 1} = \frac{120a^2}{(a^4 - 16)(a^4 - 1)}. \quad (3)$$

$$50. \frac{2}{a + b} - \frac{2}{a - b} + \frac{4b}{a^2 - b^2}.$$

$$51. \frac{1 + x}{1 - x} - \frac{1 - x}{1 + x} + \frac{1 + x^2}{1 - x^2}.$$

$$52. \frac{3p + 4q}{3p - 4q} + \frac{3p - 4q}{3p + 4q} - \frac{24pq}{9p^2 - 16q^2}.$$

$$53. \frac{1}{a+1} + \frac{1}{a-1} + \frac{2a}{a^2+1} - \frac{a^3}{a^4-1}.$$

$$54. \frac{2}{x+y} + \frac{3}{x-y} - \frac{5}{x^2-y^2}.$$

$$55. \frac{1}{a+b} + \frac{4b}{a^2-b^2} - \frac{1}{a-b} - \frac{2b}{a^2+b^2}.$$

$$56. \frac{a}{(a-b)(a-c)} + \frac{b}{(b-a)(b-c)} + \frac{c}{(c-a)(c-b)}.$$

SOLUTION. In examples of this type it is advisable to arrange all factors in the denominator in a similar manner; here in alphabetical order.

Instead of $(b-a)$, write $-(a-b)$.

Instead of $(c-a)$, write $-(a-c)$.

Instead of $(c-b)$, write $-(b-c)$.

Substituting the values and considering § 144, the expression becomes

$$\frac{a}{(a-b)(a-c)} - \frac{b}{(a-b)(b-c)} + \frac{c}{(a-c)(b-c)}.$$

The L. C. D. is $(a-b)(b-c)(a-c)$.

Hence the expression equals

$$\frac{a(b-c) - b(a-c) + c(a-b)}{(a-b)(b-c)(a-c)} = 0.$$

$$57. \frac{2}{1-x} + \frac{3}{x-1}.$$

$$60. \frac{3}{2a-3} - \frac{2}{3-2a} + \frac{15}{9-4a^2}.$$

$$58. \frac{6a}{a-b} + \frac{6b}{b-a}.$$

$$61. \frac{x}{3+x} - \frac{x}{3-x} - \frac{x^2}{x^2-9}.$$

$$59. \frac{x^2}{x^2-1} + \frac{x}{x+1} - \frac{x}{1-x}.$$

$$62. a + \frac{1}{a+1} - \frac{1}{1-a} + \frac{1}{a-1}.$$

$$63. a^2 + a + 1 + \frac{a^3}{1-a} - \frac{3}{a-1}.$$

$$64. \frac{x^2+1}{x-1} - (x-1) - \frac{x^2-1}{1-x} + \frac{x^3}{x^2-1}.$$

$$65. \frac{1}{(a-b)(a-c)} + \frac{1}{(b-c)(b-a)} + \frac{1}{(c-a)(c-b)}.$$

$$66. \frac{a}{(a-b)(a-c)} + \frac{b}{(b-a)(b-c)} + \frac{c}{(c-a)(c-b)}$$

$$67. \frac{x+3}{x^2+3x+2} - \frac{x-2}{3+2x-x^2}$$

$$68. \frac{x-1}{x^2+3x+2} + \frac{x+1}{6+x-x^2} - \frac{x-2}{x^2-2x-3}$$

$$69. 2x+1 - \frac{3x^3+2x^2-1}{2x-1}$$

$$70. 2x + \frac{3x^4-8x^3+1}{4x^2-4x+1} - 1.$$

152. Reducing mixed expressions to improper fractions. Since an integral expression may be written in the fractional form with the denominator 1, this is merely a special case of addition of fractions. *E.g.* $a + \frac{b}{c} = \frac{a}{1} + \frac{b}{c} = \frac{ac+b}{c}$. Hence the numerator of the required fraction is obtained by multiplying the integral expression by the denominator of the fraction and adding (or subtracting) the numerator of the given fraction.

EXERCISE 61

Reduce the following expressions to fractional form :

$$1. a - \frac{b}{c}$$

$$7. a - 3 + \frac{5a-6}{a-2}$$

$$2. \frac{x+1}{x-1} + 1.$$

$$8. a^2 - a + 1 - \frac{a+1}{a-1}$$

$$3. \frac{y^2}{x-y} + x + y.$$

$$9. a - 1 - \frac{a+1}{a^2-a+1}$$

$$4. a - 3 - \frac{a^2+6}{a-2}$$

$$10. a^2 + a + 1 - \frac{a^3-a+1}{a-1}$$

$$5. 5a - \frac{3ab-7b}{2b}$$

$$11. 3x - y - \frac{10xy}{x-3y}$$

$$6. a^2 - 1 - \frac{a^2-1}{a-1}$$

$$12. \frac{a^2-2b^2}{2a-3b} + a - 3b.$$

153. To reduce a fraction to an integral or mixed expression.

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \quad (\S\ 74)$$

Hence
$$\frac{5a^2 - 15a - 7}{5a} = \frac{5a^2}{5a} - \frac{15a}{5a} - \frac{7}{5a} = a - 3 - \frac{7}{5a}.$$

Ex. 1. Reduce $\frac{4x^3 - 2x^2 + 4x - 17}{2x - 3}$ to a mixed expression.

$$\begin{array}{r|l} 4x^3 - 2x^2 + 4x - 17 & 2x - 3 \\ \hline 4x^3 - 6x^2 & 2x^2 + 2x + 5 \\ \hline 4x^2 + 4x & \\ 4x^2 - 6x & \\ \hline + 10x - 17 & \\ + 10x - 15 & \\ \hline - 2 & \end{array}$$

Therefore
$$\frac{4x^3 - 2x^2 + 4x - 17}{2x - 3} = \frac{(2x^2 + 2x + 5)(2x - 3) - 2}{(2x - 3)}$$

$$= 2x^2 + 2x + 5 - \frac{2}{2x - 3}.$$

A fraction may be reduced by division to an integral or mixed expression, if the degree of the numerator is equal to, or greater than, that of the denominator.

If the remainder be a polynomial whose first term is negative, it is customary to write a minus sign before the fractional part of the result and to change all the signs of its numerator

E.g.
$$\frac{2x^3 + 4x^2 - 3x + 3}{x^3 + 3x^2 - 4x + 3} = 2 + \frac{-2x^2 + 5x - 3}{x^3 + 3x^2 - 4x + 3}$$

$$= 2 - \frac{2x^2 - 5x + 3}{x^3 + 3x^2 - 4x + 3}.$$

EXERCISE 62

Reduce each of the following fractions to a mixed or integral expression:

1. $\frac{6x^2 - 9x + 8}{3x}$

2. $\frac{6x^3 + 8x^2 - 4x - 7}{2x^2}$

3. $\frac{x^2 - 7x + 4}{x - 3}$.

7. $\frac{a^3 + 3a^2b + 3ab^2 + 2}{a + b}$.

4. $\frac{x^2 + 8x + 13}{x + 2}$.

8. $\frac{2x^2 - 6x + 17}{2x - 4}$.

5. $\frac{x^2 + 1}{x + 1}$.

9. $\frac{a^3 + b^3 + 2}{a + b}$.

6. $\frac{a^3 + 8b^3}{a - 2b}$.

10. $\frac{a^3 - 4}{a^2 + a + 1}$.

MULTIPLICATION OF FRACTIONS

154. Fractions are multiplied by taking the product of the numerators for the numerator, and the product of the denominators for the denominator ; or, expressed in symbols :

If $x = \frac{a}{b}$ and $y = \frac{c}{d}$,

then $xy = \frac{ac}{bd}$.

According to § 139, this really means :

If $bx = a$ and $dy = c$,

then $bdxy = ac$.

But in this form the correctness of the conclusion is obvious.
(§ 90, No. 3.)

Hence it is proved that

$$xy = \frac{ac}{bd}, \text{ or since } x = \frac{a}{b} \text{ and } y = \frac{c}{d},$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

155. Since $\frac{a}{1} = a$, we may extend any principle proved for fractions to integral numbers, e.g. $\frac{a}{b} \times c = \frac{ac}{b}$.

To multiply a fraction by any integer, multiply the numerator by that integer.

156. *Common factors in the numerators and the denominators should be canceled before performing the multiplication. (In order to cancel common factors, each numerator and denominator has to be factored.)*

Ex. 1. Simplify $\frac{4a^3b}{3cd^5} \times \frac{6c^2d^2}{5ax} \times -\frac{7x^4y}{2ab^2}$.

$$\begin{aligned}\text{The expression} &= \frac{-4 \cdot 6 \cdot 7 a^3 b c^2 d^2 x^4 y}{3 \cdot 5 \cdot 2 \cdot a \cdot a b^2 c d^5 x} \\ &= -\frac{28 a c x^3 y}{5 b d^3}.\end{aligned}$$

Ex. 2. Simplify $\frac{a^3 - b^3}{a^2 - 5ab + 6b^2} \cdot \frac{a^3 - 4ab^2}{3a^2 + 3ab + 3b^2} \cdot \frac{6a - 18b}{7(a-b)^2}$.

$$\begin{aligned}&= \frac{(a-b)(a^2 + ab + b^2)}{(a-2b)(a-3b)} \cdot \frac{a(a+2b)(a-2b)}{3(a^2 + ab + b^2)} \cdot \frac{6(a-3b)}{7(a-b)^2} \\ &= \frac{2a(a+2b)}{7(a-b)}.\end{aligned}$$

EXERCISE 63

Find the following products:

1. $\frac{m^3 n^4}{36} \cdot \frac{24}{m^4 n^3}$.

4. $\frac{x^2 y^3}{7 a z^3} \cdot \frac{7 a^2 z}{4 x^2 z}$.

2. $\frac{ab^2 c^3}{2 a^4} \cdot \frac{3 d^3}{5 a^2 b^3}$.

5. $\frac{3 x^2 y^3}{8 z^4} \cdot \frac{6 y^2 z^3}{5 x^5} \cdot \frac{10 a^2 z^2}{9 y^2}$.

3. $\frac{14 x^2 y^3}{15 z^2} \cdot \frac{20 z^3}{21 x^3 y^3}$.

6. $\frac{12 m + 12 n}{24 u + 24 r} \cdot \frac{u + r}{m + n}$.

7. $\frac{4 a^2 - 9 b^2}{2 a + 3 b} \cdot \frac{1}{2 a - 3 b}$.

8. $\frac{4 a^2 + 12 ab + 9 b^2}{3 a^2} \cdot \frac{4 a^3}{(2 a + 3 b)^2}$.

9. $\frac{a^2 - 169}{a^2 - 64} \cdot \frac{a^2 - 8 a}{a - 13}$.

10. $\frac{5 a}{a^2 - 5 a + 6} \cdot \frac{a^2 - 6 a + 9}{5 a^2 + 5 a}$.

11. $\frac{a+1}{a+2} \cdot \frac{a+2}{a+3}$ 13. $\frac{x^2-5x+4}{x^2+5x+4} \cdot \frac{x^2+3x-4}{x^2-3x-4}$
12. $\frac{x^2-1}{x+3} \cdot \frac{x^2-4x+3}{x^2+x} \cdot \frac{7}{(x-1)^2}$ 14. $\frac{a^2b+ab^2}{a^2b-ab^2} \cdot \frac{a^3-ab^2}{a^3+ab^2}$
15. $\frac{1}{(x-1)^2} \cdot \frac{x^2-6x+5}{6x+6} \cdot \frac{5x^2-5}{x}$
16. $\frac{a^2+18a+80}{a^2+5a-50} \cdot \frac{a^2+6a-7}{a^2+15a+56} \cdot \frac{a-5}{a-1}$
17. $\frac{a^4-8a}{2b^2-2} \cdot \frac{4b^2-8b+4}{a^2-2a} \cdot \frac{b+2}{a^2+2a+4}$
18. $\left(\frac{a+1}{a-1}\right)^2$ 20. $\left(\frac{a-b}{a+b}\right)^2 \cdot \frac{a^2+ab}{ab-b^2}$
19. $\left(\frac{a-1}{a+2}\right)^2 \cdot \frac{a^2-4}{a^2-1}$ 21. $\left(\frac{4}{x+1}\right)^2 \cdot \frac{(x+1)^3}{x-7} \cdot \frac{x^2-49}{16}$

DIVISION OF FRACTIONS

157. Divide $\frac{a}{b}$ by $\frac{c}{d}$.

Let

$$x = \frac{a}{b} \div \frac{c}{d}$$

Then, by definition of division, $x \cdot \frac{c}{d} = \frac{a}{b}$.

Multiplying both members by $\frac{d}{c}$, $x = \frac{a}{b} \cdot \frac{d}{c}$.

Or

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$$

158. To divide an expression by a fraction, invert the divisor and multiply it by the dividend. Integral or mixed divisors should be expressed in fractional form before dividing.

159. The reciprocal of a number is the quotient obtained by dividing 1 by that number.

The reciprocal of a is $\frac{1}{a}$.

The reciprocal of $\frac{3}{4}$ is $1 \div \frac{3}{4} = \frac{4}{3}$.

The reciprocal of $\frac{a+b}{x}$ is $1 \div \frac{a+b}{x} = \frac{x}{a+b}$.

Hence the reciprocal of a fraction is obtained by inverting the fraction, and the principle of division may be expressed as follows:

160. To divide an expression by a fraction, multiply the expression by the reciprocal of the fraction.

Ex. 1. Divide $\frac{x^3 - y^3}{x^3 + xy^2}$ by $\frac{x^2 - y^2}{x^2y + y^3}$.

$$\begin{aligned} \frac{x^3 - y^3}{x^3 + xy^2} \div \frac{x^2 - y^2}{x^2y + y^3} &= \frac{x^3 - y^3}{x^3 + xy^2} \times \frac{x^2y + y^3}{x^2 - y^2} \\ &= \frac{(x - y)(x^2 + xy + y^2)}{x(x^2 + y^2)} \times \frac{y(x^2 + y^2)}{(x + y)(x - y)} \\ &= \frac{y(x^2 + xy + y^2)}{x(x + y)}. \end{aligned}$$

EXERCISE 64

Simplify the following expressions:

1. $\frac{4a^3b}{3x^2y} \div \frac{4ab^3}{3xy}$

5. $\frac{a^2 + ab}{a^2 - ab} \div \frac{a^3 + a^2b}{ab^2 + b^3}$

2. $\frac{17a^4b^4}{7xy^2} \div \frac{3a^4b^3}{14x^2y^2}$

6. $\frac{a^2 - 3a + 2}{a^2 - 5a + 4} \div \frac{(a - 2)^2}{(a - 1)^2}$

3. $\frac{9x^2}{y^2} \div x^2$

7. $\frac{4a^3 - 9ab^2}{2a + 3b} \div 2a - 3b$

4. $\frac{9a}{b^4} \div 7$

8. $\frac{a + 5b}{a^2 + 6ab} \div \frac{ab + 5b^2}{a^3 + 6a^2b}$

9. $\frac{x^2 + 14x - 15}{x^2 + 4x - 5} \div \frac{x^2 + 12x - 45}{x^2 + 6x - 27}.$
10. $\frac{a^2 - b^2}{(a + b)^2} \div \frac{(a - b)^2}{2a}.$
11. $\frac{a^2 - 5a}{a - 1} \times \frac{a^2 + a - 20}{a^2 - 25} \div \frac{a^2 - 2a - 8}{a^2 + a - 2}.$
12. $\frac{a^3 - b^3}{a^3 + a^2b} \div \frac{a^2 + ab + b^2}{ab^2 + a^2b}.$
13. $\frac{2a^2 - 13a + 15}{4a^2 - 9} \times \frac{2a + 1}{2a - 1} \div \frac{a - 5}{2a - 1}.$
14. $\frac{3b^2}{a^2 - ab} \times \frac{a(a + 2b)}{ab + b^2} \div \frac{2ab}{a^2 - b^2}.$
15. $\frac{ab}{a + b} \times \frac{6a^2b - 4ab^2}{45a^2 - 20b^2} \div \frac{a^2b^2}{15a + 10b}.$
16. $\frac{6a^2 - a - 2}{4a^2 + 4a - 3} \times \frac{2a^2 - 5a - 12}{9a^2 + 6a - 8} \div \frac{2a^2 - 7a - 4}{6a^2 + 5a - 4}.$
17. $\frac{a^2 - 2a + 4}{a - 5} \times \frac{a^2 + a - 2}{a^2 - 2a + 1} \div \frac{a^4 + 8a}{a^3 + 4a^2 - 5a}.$
18. $\frac{(a + b)^2}{a - 3b} \div \frac{a^2 - b^2}{a^3 - 27b^3}.$
19. $\frac{x + y}{x^3 - y^3} \times \frac{x^3 + x^2y + xy^2}{(x + y)^3} \div \frac{x^2 - 3xy + 2y^2}{(x + y)^2}.$
20. $\frac{x^2 + xy}{x^2 - xy} \times \frac{x^2 + xy + y^2}{x + y} \div \frac{x^3 + x^2y + xy^2}{x - y}.$
21. $\frac{49x^2y^2 - z^4}{a^2 - 4} \times \frac{(a - 2)^3}{7xy + z^2} \div \frac{(a - 2)^2}{(a + 2)^2}.$
22. $\frac{a^2 - (b - c)^2}{c^2 - (b - a)^2} \div \frac{(b - c)^2 - a^2}{(a - b)^2 - c^2}.$

COMPLEX FRACTIONS

161. A **complex fraction** is a fraction whose numerator or denominator, or both, are fractional.

Ex. 1. Simplify $\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{\frac{a}{b} - \frac{b}{c} + \frac{c}{a}}.$

$$\begin{aligned} \frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{\frac{a}{b} - \frac{b}{c} + \frac{c}{a}} &= \frac{\frac{a^2c + ab^2 + bc^2}{abc}}{\frac{a^2c - ab^2 + bc^2}{abc}} \\ &= \frac{a^2c + ab^2 + bc^2}{abc} \times \frac{abc}{a^2c - ab^2 + bc^2} \\ &= \frac{a^2c + ab^2 + bc^2}{a^2c - ab^2 + bc^2}. \end{aligned}$$

162. In many examples the easiest mode of simplification is to multiply both the numerator and the denominator of the complex fraction by the L. C. M. of their denominators.

If the numerator and denominator of the preceding examples are multiplied by abc , the answer is directly obtained.

Ex. 2. Simplify $\frac{\frac{x+y}{x-y} + \frac{x-y}{x+y}}{\frac{1}{x-y} - \frac{1}{x+y}}.$

Multiplying the terms of the complex fraction by $(x+y)(x-y)$, the expression becomes

$$\begin{aligned} \frac{(x+y)^2 + (x-y)^2}{(x+y) - (x-y)} &= \frac{2x^2 + 2y^2}{2y} \\ &= \frac{x^2 + y^2}{y}. \end{aligned}$$

EXERCISE 65

Simplify:

1. $\frac{\frac{a}{b}}{\frac{c}{c}}$

2. $\frac{\frac{a}{b}}{\frac{c}{c}}$

3. $\frac{\frac{a}{b}}{\frac{c}{d}}$

4. $\frac{a + \frac{b}{c}}{d}$

13. $\frac{\frac{2a^2}{26} - 2b}{\frac{2a}{26} - 1}$

14. $\frac{\frac{1}{a+1} + \frac{1}{a-1}}{\frac{1}{a-1} - \frac{1}{a+1}}$

15. $\frac{\frac{\frac{x}{y} - \frac{y}{x}}{\frac{x}{y} + \frac{y}{x} + 2}}$

16. $\frac{\frac{\frac{x}{y} - \frac{y}{x}}{\frac{x}{y} + \frac{y}{x} - 2}}$

5. $\frac{\frac{a}{b}}{b - \frac{c}{d}}$

6. $\frac{a + \frac{b}{c}}{a - \frac{b}{c}}$

7. $\frac{\frac{a}{b} - \frac{c}{d}}{\frac{a}{b} + \frac{c}{d}}$

8. $\frac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{c} - \frac{1}{a}}$

9. $\frac{\frac{a+b}{c}}{\frac{1}{c} + \frac{1}{a}}$

10. $\frac{a + \frac{1}{a}}{a - \frac{1}{a}}$

11. $\frac{\frac{a}{5} + 4x}{\frac{a}{5} + 5x}$

12. $\frac{3a - \frac{3x^2}{3a}}{1 + \frac{2x}{a}}$

17. $\frac{\frac{x+1}{x-1} - \frac{x-1}{x+1}}{\frac{x+1}{x-1} + \frac{x-1}{x+1}}$

18. $\frac{\frac{2a-5b}{2a+5b} + \frac{2a+5b}{2a-5b}}{\frac{2a+5b}{2a-5b} - \frac{2a-5b}{2a+5b}}$

19. $\frac{\frac{a^2}{x^2} + \frac{a}{x} + 1}{\frac{b^2}{x^2} - \frac{b}{x} + 1}$

20. $\frac{3x^2 + x - 2}{1 - \frac{2}{3x}}$

$$21. \frac{\frac{x}{y} - \frac{y}{x}}{1 - \left(\frac{x}{y}\right)^2}.$$

$$26. \frac{a}{1 + \frac{a}{1 + \frac{1}{a}}}.$$

$$22. \frac{\frac{1}{x-2} - \frac{1}{x-3}}{1 + \frac{1}{x^2 - 5x + 6}}.$$

$$27. 2 + \frac{1}{5 + \frac{1}{6 + \frac{1}{3}}}.$$

$$23. \frac{(1+x) \left\{ \frac{1}{x} - \frac{1}{1+x} \right\}}{\frac{4}{x} + x}.$$

$$28. \frac{a}{b + \frac{1}{c + \frac{1}{d}}}.$$

$$24. \frac{\frac{x}{y} + \frac{y}{x}}{\frac{x}{y} - \frac{y}{x}} + \frac{\frac{x}{y} - \frac{y}{x}}{\frac{x}{y} + \frac{y}{x}}.$$

$$29. \frac{1}{1 + \frac{1}{1 + \frac{1}{a}}}.$$

$$25. \frac{x + \frac{2}{x-3}}{x - \frac{2}{x-2}} \times \frac{x-2 + \frac{1}{x}}{x-3 + \frac{1}{x}}.$$

$$30. \frac{1}{a^2 - \frac{a+1}{1 + \frac{1}{a-1}}}.$$

REVIEW EXERCISE II

Simplify:

$$1. \left(a + b\right) \left(\frac{1}{a} + \frac{1}{b}\right).$$

$$4. \left(\frac{5x}{12y} - \frac{15}{16}\right) \left(\frac{4}{5}y - \frac{12x^2}{25y^2}\right).$$

$$2. \left(\frac{5a}{6b} - 3a\right) \div \left(\frac{4a}{15b} - 2b\right).$$

$$5. \left(x - y\right) \left(\frac{1}{x} - \frac{1}{y}\right).$$

$$3. \left(\frac{a}{b} - \frac{c}{d}\right) (ad + bc).$$

$$6. \left(\frac{a}{b} - 1\right) \left(\frac{b}{a} + 1\right).$$

$$7. \left(\frac{x}{a} - \frac{y}{b}\right) \left(\frac{a}{x} - \frac{b}{y}\right).$$

$$8. \left(\frac{2a}{3b} - \frac{3b}{2a}\right) \div \left(-6ab - \frac{5}{12ab}\right).$$

9. $\left(2 - \frac{x}{x+y}\right)\left(1 + \frac{2y}{x-y}\right).$
10. $\left(\frac{2x}{5y}\right)^3.$
11. $\left(\frac{2x}{5y} + \frac{5y}{4x}\right)^3.$
12. $\left(\frac{3ab}{2c} - \frac{2bc}{9a}\right)^2.$
13. $\left(\frac{a}{b} + \frac{b}{a}\right)^2 + \left(\frac{a}{b} - \frac{b}{a}\right)^2.$
14. $\left(\frac{2a}{3b} - \frac{4b}{3a}\right)^2 + \left(\frac{2a}{4b} - \frac{2b}{3a}\right)^2.$
15. $\left(\frac{x}{4y} + \frac{5y}{x}\right)^2 + \left(\frac{x}{4y} - \frac{5y}{x}\right)^2.$
16. $\left(\frac{p}{5q} + \frac{3q}{2p}\right)^2 - \left(\frac{2p}{3q} - \frac{5q}{2p}\right)^2.$
17. $\left(\frac{a^3b^3}{c^3} + \frac{a^2b^2}{c^2} + \frac{ab}{c} + 1\right)\left(\frac{ab}{c} - 1\right).$
18. $\left(\frac{27x^3y^3}{64z^3} - \frac{9x^2y^2}{16z^2} + \frac{3xy}{4z} - 1\right)\left(\frac{3xy}{4z} + 1\right).$
19. $\left(\frac{x^2}{y^2} + \frac{x}{y} + 1\right)\left(\frac{x^2}{y^2} - \frac{x}{y} + 1\right).$
20. $\left(\frac{a^2}{b^2} + \frac{ax}{by} + \frac{x^2}{y^2}\right)\left(\frac{a}{b} - \frac{x}{y}\right).$
21. $\left(\frac{5a^4}{8b^2} + 12ab - \frac{9b^4}{16a^2}\right) \div \frac{15a^2}{32b^2}.$
22. $\left(\frac{3a+3b}{3a-4b} - \frac{2a-3b}{3a+4b}\right) \div \frac{4a^2-9b^2}{9a^2-16b^2}.$
23. $\left(\frac{a}{a+b} - \frac{a^2}{(a+b)^2} + \frac{b}{a-b} - \frac{b^2}{a^2-b^2}\right) \div \frac{2ab}{a^2-b^2}.$
24. $\left(x^2 - \frac{1}{9y^2}\right) \div \left(x - \frac{1}{3y}\right).$
25. $\left(\frac{4a^2}{9b^2} - \frac{4c^2}{d^2}\right) \div \left(\frac{2a}{3b} + \frac{2c}{d}\right).$
26. $\left(a^3 - \frac{1}{a^3}\right) \div \left(a - \frac{1}{a}\right).$
27. $\left(a^3 + \frac{1}{64}\right) \div \left(a + \frac{1}{4}\right).$
28. $\left(x^3 - \frac{1}{x^3}\right) \div \left(x^2 + 1 + \frac{1}{x^2}\right).$
29. $\frac{\frac{2xy}{3} - \frac{y^2}{4}}{\frac{5x}{6} - \frac{8y}{9}} \times \frac{\frac{5}{6yz} - \frac{8}{9xz}}{\frac{2}{y} - \frac{3}{4x}}.$
30. $\left(\frac{x+1}{x-1}\right)^2 - \left(\frac{x-1}{x+1}\right)^2.$
31. $[1 \div (x-1)] + [1 \div (x+1)].$
32. $\frac{1+5x+5x^2+x^3}{1+x^3}.$

$$33. \frac{a}{4} \left(\frac{2}{a-b} - \frac{2}{a+b} \right) \times \frac{a^2 - b^2}{a^3b + ab^3} \div \frac{1}{a^2 + b^2}.$$

$$34. \frac{x+y}{(y-z)(y-x)} + \frac{y+z}{(z-x)(z-y)} + \frac{z+x}{(x-y)(x-z)}.$$

$$35. \frac{2}{a} - \left[\frac{2a-3}{4a^2-1} - \frac{3}{1-2a} - \frac{1}{1+2a} \right].$$

$$36. \frac{x-2}{6} - \left[\frac{x-4}{9} - \left(\frac{2-3x}{4} - \frac{2x+1}{12} \right) \right].$$

$$37. \frac{a^5 + a^4b - a^2b^3 - ab^4}{a^4b - a^3b^2 + ab^4 - b^5}.$$

$$38. 2 \left[\frac{1}{x-3} - \frac{1}{x-2} \right] - \frac{2}{x^2 - 5x + 6}.$$

$$39. \frac{1}{x - \frac{1}{x}} + \frac{1}{1 + \frac{1}{x}} - \frac{2}{1 - \frac{1}{x^2}}.$$

40. If $a=3$, $b=4$, $c=5$, $d=2$, find the value of

$$\frac{a}{b + \frac{1}{c + \frac{1}{d}}}.$$

Factor the following expressions:

$$41. 9 - \frac{6x}{y} + \frac{x^2}{y^2}.$$

$$42. \frac{x^2}{y^2} - 3\frac{x}{y} + 2.$$

$$43. \frac{x^2}{y^2} - \frac{8ax}{by} + \frac{16a^2}{b^2}.$$

$$44. \frac{x^3}{y^3} - \frac{1}{27}.$$

$$45. \frac{64x^3}{c^3} + 1.$$

46. Simplify $\left[\frac{4}{5}x - \frac{2}{3}y - \left(\frac{7}{15}x - \frac{1}{6}y \right) \right] \div a \left(\frac{1}{3}x - \frac{1}{2}y \right).$

$$47. \text{Simplify } \frac{\frac{x-y}{3} - \frac{x-y}{4}}{5} - \frac{\frac{x-y}{5} - \frac{x-y}{6}}{3} + \frac{8x-y}{15}.$$

48. Divide $\frac{1}{9} - \frac{1}{25}x^2 + \frac{2}{35}x^3 - \frac{1}{49}x^4$ by $\frac{1}{3} + \frac{1}{5}x - \frac{1}{7}x^2$.

49. From $\frac{a}{b}x^n - \frac{c}{d}y^p + \frac{e}{f}z^q$

subtract $\frac{a}{3b}x^n - \frac{c}{2d}y^p + \frac{e}{f}z^q.$

50. Simplify $a^{x-y} \cdot a^{y-z} \cdot a^{z-2x} \cdot a^{x+1}.$

51. A man left half his property to his wife, one third to his daughter, and the remainder to his son. If the son received \$1700, how many dollars did the man leave?

52. A man bought five pounds of coffee of two different kinds for \$1.55. If the better kind cost 35¢ and the poorer kind 25¢ per pound, how many pounds did he buy of each?

53. A and B set out walking in the same direction, but A has a start of 2 miles. A walks at the rate of 3 miles, and B at the rate of $3\frac{1}{2}$ miles, per hour. After how many hours will B overtake A, and how far will he have walked?

54. The sum of 13 dollars is made up of dollars, half dollars, and quarters. If the number of dollars is half the number of quarters and four times the number of half dollars, find the number of each.

CHAPTER IX

FRACTIONAL AND LITERAL EQUATIONS

FRACTIONAL EQUATIONS

163. Clearing of fractions. If an equation contains fractions, these may be removed by multiplying each term by the L. C. M. of the denominator.

Ex. 1. Solve $\frac{2x}{6} - \frac{x-3}{3} = 12 - \frac{x+4}{2} - x$.

Multiplying each term by 6 (Axiom 3, § 90),

$$2x - 2(x-3) = 72 - 3(x+4) - 6x.$$

Removing parentheses, $2x - 2x + 6 = 72 - 3x - 12 - 6x$.

Transposing, $2x - 2x + 3x + 6x = -6 + 72 - 12$.

Uniting, $9x = 54$.

$$x = 6.$$

Check. If $x = 6$, each member is reduced to 1.

Ex. 2. Solve $\frac{5}{x-1} - \frac{14}{x+1} = -\frac{9}{x+3}$.

Multiplying by $(x-1)(x+1)(x+3)$,

$$5(x+1)(x+3) - 14(x-1)(x+3) = -9(x+1)(x-1).$$

Simplifying, $5x^2 + 20x + 15 - 14x^2 - 28x + 42 = -9x^2 + 9$.

Transposing, $5x^2 - 14x^2 + 9x^2 + 20x - 28x = -15 - 42 + 9$.

Uniting, $-8x = -48$,

$$x = 6.$$

Check. If $x = 6$, each member is reduced to -1.

EXERCISE 66

Solve the following equations:

1. $\frac{x-2}{3} = \frac{3-x}{2}$.
2. $\frac{x-3}{4} = \frac{4-x}{3}$.
3. $\frac{9x-5}{4} + 4x = \frac{284-x}{5}$.
4. $\frac{13-x}{3} + \frac{x}{2} = \frac{12-x}{4}$.
5. $\frac{19-x}{2} + x = \frac{41-x}{3}$.
6. $\frac{x-2}{2} + \frac{x+2}{6} = \frac{4-4x}{12}$.
7. $\frac{x-3}{20} + \frac{x-4}{15} = \frac{5-x}{12}$.
8. $\frac{5x-4}{3} = x+4$.
9. $\frac{3x+10}{7} = \frac{2x+4}{5}$.
10. $\frac{x+3}{x-3} = 3$.
11. $\frac{26}{x} = 13$.
12. $\frac{45}{4x} = 15$.
13. $\frac{45}{x} + 7 = 12$.
14. $\frac{65}{3x+2} = 13$.
15. $\frac{150}{3x+1} - 2 = 13$.
16. $\frac{16}{x-3} = 4$.
17. $\frac{4}{x} = \frac{5}{x} - 1$.
18. $\frac{1}{x} + \frac{2}{x} + \frac{3}{x} = 1$.
19. $\frac{4}{5x} - \frac{7}{10x} = \frac{1}{10}$.
20. $\frac{42}{x} - \frac{1}{x} + \frac{1}{3x} = 20$.
21. $\frac{x-2}{x+3} = \frac{3}{4}$.
22. $\frac{12+x}{2x} = \frac{12+2x}{3x}$.
23. $\frac{4x-1}{5} - 2 = \frac{3}{2}x - 3$.
24. $\frac{6x-5}{4} - 2 = \frac{5x}{3} - 5$.
25. $\frac{5x}{2} - \frac{4x}{3} + \frac{3x}{4} - \frac{2x}{5} + \frac{x}{6} = 101$.
26. $\frac{x+3}{x} = \frac{x+9}{x+4}$.
27. $\frac{20-x}{x-8} = \frac{x-6}{15-x}$.
28. $\frac{x+1}{x-4} = \frac{x}{x-3}$.
29. $\frac{2x+4}{3x-5} = \frac{5}{2}$.
30. $\frac{8x^2-3x+4}{12x^2+5x-3} = \frac{2}{3}$.

$$31. \frac{2(5x-9)}{3} - x + 14 = \frac{3(7x-19)}{4} - 4.$$

$$32. \frac{7(5x+2)}{6} - \frac{3(7x-4)}{5} = \frac{2(9x+2)}{5}.$$

$$33. \frac{9}{13}(2x-9) + \frac{1}{5}(x-4) = \frac{7}{5}(3x-2) - (3x-5).$$

$$34. \frac{8x-7}{6x+7} = \frac{12x-19}{9x-2}.$$

$$38. \frac{7}{2x-5} = \frac{13}{6x^2-15x} - \frac{4}{3x}.$$

$$35. \frac{3}{x+1} - \frac{2}{x-2} = \frac{1}{x+2}.$$

$$39. \frac{15}{3x-2} - \frac{20}{3x^2-2x} = \frac{10}{3x-2}.$$

$$36. \frac{6}{x+2} + \frac{2}{x-2} = \frac{16}{x^2-4}.$$

$$40. \frac{51}{2x+3} - \frac{23x+26}{4x^2-9} = \frac{22}{2x-3}.$$

$$37. \frac{7}{x+3} + \frac{1}{x-3} = \frac{24}{x^2-9}.$$

$$41. \frac{4}{3x+2} + \frac{2}{3x-2} = \frac{15x+2}{9x^2-4}.$$

$$42. \frac{63}{4x^2-2x} - \frac{10}{6x^2+3x} = \frac{66}{4x^2-1}.$$

$$43. \frac{4x-7}{6x+18} = \frac{7x+2}{10x+30} - \frac{11}{45}.$$

$$44. \frac{19}{3x-2} - \frac{5}{5x-6} = \frac{53x+4}{(3x-2)(5x-6)}.$$

$$45. \frac{5}{x-7} - \frac{7}{x+3} = \frac{5x+1}{x^2-4x-21}.$$

$$47. \frac{2x+5}{x-2} - \frac{x+29}{x+9} = 1.$$

$$46. \frac{7}{x+4} - \frac{3}{x-5} = \frac{26x-25}{x^2-x-20}.$$

$$48. \frac{6x-11}{x-11} - \frac{3x-2}{x+1} = 3.$$

$$49. \frac{5x-1}{2x+1} + \frac{2x+9}{2x-1} = \frac{14x^2+49}{4x^2-1}.$$

$$50. \frac{3x-4}{5x-6} + \frac{2x-5}{5x+6} = \frac{25x^2-27x-42}{25x^2-36}.$$

$$51. \frac{6x+7}{2x+3} - \frac{8x-5}{3x-4} = \frac{2x^2-35x+77}{6x^2+x-12}.$$

$$52. \frac{2x-9}{3x-1} + \frac{2x-7}{2x+5} = \frac{(5x-13)(2x-4)}{6x^2+13x-5}.$$

$$53. \frac{4}{x-3} - \frac{3}{x-5} = \frac{1}{x}.$$

$$56. \frac{4}{2x+3} + \frac{3}{3x-1} = \frac{3}{x}.$$

$$54. \frac{5}{x+7} - \frac{3}{x-5} = \frac{2}{x}.$$

$$57. \frac{1}{2x-9} + \frac{6}{3x-1} = \frac{10}{4x-8}.$$

$$55. \frac{1}{x-1} + \frac{1}{x-5} = \frac{2}{x-4}.$$

$$58. \frac{9}{5x} - \frac{8}{10x-5} = \frac{4x-1}{4x^2-1}.$$

$$59. \frac{2x-1}{6x^2-4x} - \frac{1-3x}{9x^2+6x} = \frac{2}{3x}.$$

$$60. \frac{5x-.4}{.3} + \frac{1.3-3x}{2} = \frac{1.8-8x}{1.2}.$$

$$61. \frac{4(13x-.6)}{5} + \frac{3(1.2-x)}{10} = \frac{9x+.2}{20} + \frac{5+11x}{4}.$$

$$62. \frac{2-2.8x}{2(x-1)} = \frac{2.1(.2+x)}{.3-1.5x}.$$

164. If two or more denominators are monomial, and the other polynomial, it is advisable first to remove the monomial denominators only, and after simplifying the resulting equation to clear of all denominators.

Ex. 1. Solve $\frac{16x+3}{10} - \frac{2x-5}{5x-1} = \frac{8x-1}{5}.$

Multiplying each term by 10, the L. C. M. of the monomial denominators,

$$16x+3 - \frac{20x-50}{5x-1} = 16x-2.$$

Transposing and uniting, $5 = \frac{20x-50}{5x-1}.$

Multiplying by $5x-1$, $25x-5 = 20x-50.$

Transposing and uniting, $5x = -45.$

Dividing, $x = -9.$

Check. If $x = -9$, each member is reduced to $-\frac{73}{5}.$

165. Frequently it is advantageous to unite terms before the clearing of fractions.

Ex. 2. Solve $\frac{x+1}{x+2} - \frac{x+4}{x+5} = \frac{x+2}{x+3} - \frac{x+5}{x+6}$.

Uniting the fractions in each member separately,

$$\frac{x^2 + 6x + 5 - x^2 - 6x - 8}{(x+2)(x+5)} = \frac{x^2 + 8x + 12 - x^2 - 8x - 15}{(x+3)(x+6)},$$

or,
$$\frac{-3}{(x+2)(x+5)} = \frac{-3}{(x+3)(x+6)}.$$

Dividing both members by -3 and clearing fractions,

$$x^2 + 9x + 18 = x^2 + 7x + 10.$$

Transposing and uniting, $2x = -8$.

Dividing, $x = -4$.

Check. $x = -4$ reduces each member to $\frac{3}{2}$.

Solve the following equations:

63. $\frac{10x+1}{5} - \frac{4x+7}{6x+11} = \frac{6x-2}{3}.$

64. $\frac{21x+7}{9} + \frac{7x-13}{6x+3} = \frac{7x+4}{3}.$

65. $\frac{24x+7}{15} - \frac{8x+1}{5} = \frac{2x-7}{7x-6}.$

66. $\frac{17x-13}{14} - \frac{34x-75}{28} = \frac{13x+7}{6x+14}.$

67. $\frac{7x-5}{5} + \frac{x-1}{2x-15} - 1\frac{1}{10} = \frac{14x-3}{10}.$

68. $\frac{18x+43}{15} - \frac{3x+5}{2x-25} = \frac{6x}{5} + 2.$

69. $\frac{7x+13}{9} + \frac{7x-29}{5x-12} = \frac{14x+19}{18} + 1\frac{1}{9}.$

70. $\frac{6}{x-5} - \frac{6}{x-3} = \frac{4}{x-7} - \frac{4}{x-4}.$

$$71. \frac{1}{x-8} - \frac{1}{x-6} = \frac{1}{x-4} - \frac{1}{x-2}. \quad 72. \frac{x-1}{x-2} - \frac{x-4}{x-5} = \frac{x-5}{x-6} - \frac{x-8}{x-9}.$$

$$73. \frac{x-4}{x-5} - \frac{x-7}{x-8} = \frac{x-5}{x-6} - \frac{x-8}{x-9}.$$

$$74. \frac{x+2}{12x^2+34x+24} - \frac{x+14}{12x^2-14x-48} + \frac{6x-4}{9x^2-12x-32} = 5.$$

$$75. \frac{\frac{2-x}{3} - \frac{1-3x}{9}}{\frac{1}{1-x} + \frac{1}{1+x}} \times \frac{9 + \frac{36}{x}}{\frac{5}{2x} - \frac{5x}{2}} = 5.$$

$$76. (4x - \frac{1}{2})(5x + \frac{3}{4}) = (2x + \frac{1}{2})(10x - \frac{3}{2}).$$

LITERAL EQUATIONS

166. Literal equations (§ 88) are solved by the same method as numerical equations.

When the terms containing the unknown quantity cannot be actually added, they are united by factoring.

Thus, $ax + bx = (a + b)x.$

$$mx + m^2x - mnx = (m + m^2 - nm)x.$$

Ex. 1. Solve $\frac{x-a}{b} + \frac{x-b}{a} = -\frac{3b+a}{a}.$

Clearing of fractions, $ax - a^2 + bx - b^2 = -3b^2 - ab.$

Transposing, $ax + bx = a^2 + b^2 - 3b^2 - ab.$

Uniting, $(a + b)x = a^2 - ab - 2b^2.$

Dividing, $x = \frac{a^2 - ab - 2b^2}{a + b}.$

Reducing to lowest terms, $x = a - 2b.$

EXERCISE 67

Solving the following equations:

1. $a + x = b - x.$

3. $a + m + x + 9 = 2m + a + 10.$

2. $a - x = b - 8.$

4. $ax = b.$

5. $mx - n = p.$
6. $a(x - b) = c.$
7. $a(b - x) = c.$
8. $4(x - a) = 3x - 5b.$
9. $3(4a - 3x) = 5(4b - x).$
10. $mx + nx = p.$
11. $m - mx = px - d.$
12. $ax + bx = ac + b.$
13. $(a + b)x = m - cx.$
14. $(a - b)x = 2a - (a + b)x.$
15. $\frac{x}{a} = b.$
16. $\frac{a}{x} = b.$
17. $\frac{a}{x} - \frac{b}{x} = c.$
18. $\frac{a}{x} - 1 = \frac{b}{x} - 9.$
19. $\frac{a - bx}{c} + b = \frac{bc - x}{c}.$
20. $\frac{x}{a} + \frac{x}{b} = c.$
21. $\frac{x + a}{b} - \frac{b}{a} = \frac{x - b}{a} + \frac{a}{b}.$
22. $a\left(b + \frac{x}{c}\right) = m\left(c - \frac{x}{b}\right).$
23. $\frac{1 + x}{1 - x} = \frac{a}{b}.$
24. $\frac{ax + b}{ax - b} = \frac{c}{a}.$
25. $\frac{x + a}{x - b} = \frac{1}{m}.$
26. $\frac{a + x}{b + 2x} = 1.$
27. $\frac{a + x}{a - x} = \frac{a + b}{a - b}.$
28. $\frac{a}{b + x} - c = d.$
29. $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = d.$
30. $\frac{a - bm}{mx} - \frac{c - bn}{nx} = 1.$
31. $\frac{a - x}{a} + \frac{b - x}{b} + \frac{c - x}{c} = 3.$
32. $\frac{x - a}{x - b} + \frac{x - c}{x - a} = 2.$
33. $\frac{x - b}{a - b} + \frac{x + b}{a + b} + \frac{ax - b^2}{a^2 - b^2} = 0.$
34. $\frac{x - 2a}{a + b} - \frac{x + 2b}{2a + 2b} = \frac{3x - 3a}{2b}.$
35. $\frac{11x}{6x + 12b} + \frac{3a(4b - 5a)}{9a^2 - 4b^2} = \frac{a}{6a - 4b}.$
36. $\frac{a(x + b)}{x + a} + \frac{b(x - a)}{x + b} = a + b.$

$$37. \frac{a}{x+a} - \frac{b}{x+b} = \frac{a-b}{x-b}.$$

$$38. \frac{a(b-x)}{bx} + \frac{b(c-a)}{cx} = \frac{a+b}{x} - \left(\frac{b}{c} + \frac{a}{b} \right).$$

$$39. \frac{5.629 + \frac{1}{x}}{5.629 - \frac{1}{x}} = \frac{(5.629)^2 + 1}{(5.629)^2 - 1}.$$

167. It frequently occurs that the unknown letter is not expressed by x , y , or z .

Ex. If $\frac{2(a-b)}{3a-c} = \frac{2a+b}{3(a-c)}$, find a in terms of b and c .

Multiplying by $3(a-c)(3a-c)$

$$6(a-b)(a-c) = (2a+b)(3a-c).$$

$$6a^2 - 6ab - 6ac + 6bc = 6a^2 - 2ac + 3ab - bc.$$

Transposing all terms containing a to one member,

$$-6ab - 6ac + 2ac - 3ab = -6bc - bc.$$

Simplifying,

$$-9ab - 4ac = -7bc.$$

Uniting the a , and multiplying by -1 , $a(9b + 4c) = 7bc$.

Dividing,

$$a = \frac{7bc}{9b + 4c}.$$

EXERCISE 68

1. If $\frac{a}{b} = \frac{c}{2}$, find a in terms of b and c .

2. If $\frac{a}{b} = c$, find b in terms of a and c .

3. If $\frac{1+a}{1-a} = \frac{b}{c}$, solve for c .

4. If $\frac{1+a}{1-a} = \frac{b}{c}$, find a in terms of b and c .

5. $\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$, solve for f .

6. Solve the same equation for p .

7. The formula for simple interest (§ 30, Ex. 3) is $i = \frac{prn}{100}$, i denoting the interest, p the principal, r the number of %, and n the number of years. Find the formula for:

(a) The principal.

(b) The rate.

(c) The time, in terms of other quantities.

8. The formula for falling bodies (§ 30, Ex. 7) is $s = \frac{1}{2} gt^2$. Find:

(a) g in terms of s and t .

(b) t^2 in terms of a and s .

9. (a) Find a formula expressing degrees of Fahrenheit (F) in terms of degrees of centigrade (C) by solving the equation $C = \frac{5}{9}(F - 32)$.

(b) Express in degrees Fahrenheit 40°C. , 100°C. , -20°C.

10. The formula for compound interest is $I = p\left(1 + \frac{r}{100}\right)^n - p$. Express p in terms of I , r , and n .

11. If C is the circumference of a circle whose radius is R , then $C = 2\pi R$. Find R in terms of C and π .

12. If $a^2 = b^2 + c^2 - 2cp$, find p in terms of a , b , and c .

13. $\frac{a}{b} = \frac{x}{c-x}$, solve for x .

14. $F = \frac{h}{2}(b+c)$. Find:

(a) h in terms of F , b , and c .

(b) b in terms of F , h , and c .

15. If $C = \frac{E}{R+r}$, find r in terms of C , E , and R .

IDENTICAL EQUATIONS

168. Identical equations may be proved by applying the same operations which are used in the solution of equations. If the resulting equations prove to be true, then the original equation is true.

Care, however, must be taken not to multiply the two members of an equation by zero, since the axiom of dividing equals by equals cannot be applied to zero divisors.

Ex. 1. Prove that $\frac{1}{a} - \frac{a+b}{2ab} = \frac{a+b}{2ab} - \frac{1}{b}$.

The identity is true,

if $2b - (a+b) = a+b - 2a$ (clearing of fractions),

or, if $b - a = b - a$ (simplifying).

But since $b - a = b - a$,

$$\frac{1}{a} - \frac{a+b}{2ab} = \frac{a+b}{2ab} - \frac{1}{b}.$$

169. Frequently a relation between letters is given in the form of an equation, and another equation has to be proved.

Ex. 2. If $ad = bc$, prove that $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

The identity $\frac{a+b}{a-b} = \frac{c+d}{c-d}$ is true,

if $(a+b)(c-d) = (a-b)(c+d)$ (clearing of fractions),

or, if $ac + bc - ad - bd = ac - bc + ad - bd$ (simplifying),

or, if $bc - ad = -bc + ad$ (canceling),

or, if $2bc = 2ad$ (transposing),

or, if $bc = ad$ (dividing).

But $bc = ad$.

Hence $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

EXERCISE 69

Prove the following identities :

$$1. \frac{a+b}{a} + \frac{b-a}{b} = \frac{b^2 + 2ab - a^2}{ab}.$$

$$2. \frac{2x^2}{x^2 - y^2} - \frac{y}{x+y} = \frac{x}{x-y} + \frac{x-y}{x+y}.$$

$$3. \frac{2a+3}{a^3-7a-6} + \frac{a+1}{a^2-a-6} = \frac{a+2}{a^2-2a-3}.$$

$$4. \frac{2a-3}{4a^3-a} + \frac{5}{2a-1} = \frac{3}{a} - \frac{2a-7}{4a^2-1}.$$

$$5. \frac{a^2}{a^2-b^2} + \frac{b}{a+b} = \frac{a}{a-b} - \frac{b^2}{a^2-b^2}.$$

If $ad = bc$, prove that :

$$6. \frac{a}{c} = \frac{b}{d}.$$

$$7. \frac{a+b}{b} = \frac{c+d}{d}.$$

$$8. \frac{a-b}{b} = \frac{c-d}{d}.$$

$$9. \frac{a-c}{a} = \frac{b-d}{b}.$$

$$10. \frac{a+2b}{a+3b} = \frac{c+2d}{c+3d}.$$

$$11. \frac{5a+3b}{2a-b} = \frac{5c+3d}{2c-d}.$$

$$12. \frac{am+bn}{ap+bq} = \frac{cm+dn}{cp+dq}.$$

$$13. \frac{a+2b-c-2d}{b-d} = \frac{c+2d}{d}.$$

$$14. \frac{3a+6b-5c-10d}{3b-5d} = \frac{a+2b}{b}.$$

If $ac = b^2$, prove that :

$$15. \frac{a}{b} = \frac{b}{c}.$$

$$16. \frac{a}{a+b} = \frac{a-b}{a-c}.$$

$$17. \frac{a^2+2b^2}{b^2+2c^2} = \frac{b^2}{c^2}.$$

If $a^2 + b^2 = c^2$, prove that :

$$18. (a-b)^2 + 2ab = c^2.$$

$$19. (a+b)^2 + (a-b)^2 = 2c^2.$$

$$20. \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1.$$

$$21. 1 + \left(\frac{a}{b}\right)^2 = \left(\frac{c}{b}\right)^2.$$

$$22. \frac{\frac{b}{c}}{1 - \frac{a}{c}} = \frac{1 + \frac{a}{c}}{\frac{b}{c}}.$$

PROBLEMS LEADING TO FRACTIONAL AND LITERAL EQUATIONS

170. Ex. 1. When between 3 and 4 o'clock are the hands of a clock together?

At 3 o'clock the hour hand is 15 minute spaces ahead of the minute hand, hence the question would be formulated: After how many minutes has the minute hand moved 15 spaces more than the hour hand?

Let x = the required number of minutes after 3 o'clock,
then x = the number of minute spaces the minute hand moves over,

and $\frac{x}{12}$ = the number of minute spaces the hour hand moves over.

Therefore $x - \frac{x}{12}$ = the number of minute spaces the minute hand moves more than the hour hand.

Or
$$x - \frac{x}{12} = 15.$$

Uniting,
$$\frac{11x}{12} = 15.$$

Multiplying by 12,
$$11x = 180.$$

Dividing,
$$x = \frac{180}{11} = 16\frac{4}{11} \text{ minutes after 3 o'clock.}$$

Ex. 2. A can do a piece of work in 3 days and B in 2 days. In how many days can both do it working together?

If we denote the required number of days by x and the piece of work by 1, then A would do each day $\frac{1}{3}$ and B $\frac{1}{2}$, while in x days they would do respectively $\frac{x}{3}$ and $\frac{x}{2}$, and hence the sentence written in algebraic symbols is $\frac{x}{3} + \frac{x}{2} = 1$.

A more symmetrical but very similar equation is obtained by writing in symbols the following sentence: "The work done by A in one day plus the work done by B in one day equals the work done by both in one day."

Let x = the required number of days.

Then $\frac{1}{x}$ = the part of the work both do in one day.

Therefore,
$$\frac{1}{3} + \frac{1}{2} = \frac{1}{x}.$$

Solving, $x = \frac{6}{5}$, or $1\frac{1}{5}$, the required number of days.

Ex. 3. The speed of an express train is $\frac{9}{5}$ of the speed of an accommodation train. If the accommodation train needs 4 hours more than the express train to travel 180 miles, what is the rate of the express train?

	TIME (hours)	RATE (miles per hour)	DISTANCE (miles)
Express train	$180 \div \frac{9x}{5} = \frac{100}{x}$	$\frac{9x}{5}$	180
Accommodation train	$180 \div x = \frac{180}{x}$	x	180

$$\text{Therefore,} \quad \frac{180}{x} = \frac{100}{x} + 4. \quad (1)$$

$$\text{Clearing,} \quad 180 = 100 + 4x.$$

$$\text{Transposing,} \quad 4x = 80.$$

$$\text{Hence} \quad x = 20.$$

$$\frac{9}{5}x = 36 = \text{rate of express train.}$$

Explanation: If x is the rate of the accommodation train, then $\frac{9x}{5}$ is the rate of the express train. But in uniform motion, $\text{Time} = \frac{\text{Distance}}{\text{Rate}}$. Hence the rates can be expressed, and the statement, "The accommodation train needs 4 hours more than the express train," gives the equation (1).

EXERCISE 70

1. Find a number whose third, fourth, and fifth parts added together make 47.

2. Find a number whose third part exceeds its fourth part by 4.

3. What number exceeds the difference of its third and fifth parts by 26? x

4. Two numbers differ by 5, and one-half the greater exceeds the smaller by 1. Find the numbers.

5. Two numbers differ by 9, and $\frac{1}{10}$ of the greater is equal to $\frac{1}{7}$ of the smaller. Find the numbers.

6. Find two consecutive numbers such that $\frac{1}{7}$ of the greater exceeds $\frac{1}{9}$ of the smaller by 1.

7. The sum of two numbers is 56, and one is $\frac{3}{5}$ of the other. What are the numbers?

8. A certain number is increased by 2, the sum multiplied by 3, the product divided by 2, the quotient increased by 2, the sum multiplied by $\frac{3}{2}$, and the result is found to be 12. What is the number?

9. Find the sum of three consecutive numbers such that one half of the first, plus one third of the second, plus one fourth of the third, equals 16.

10. A's age is one third of his age 30 years hence; how old is he?

11. Five years ago a boy's age was one third of his age six years hence. How old is he now?

12. A boy's age is one fourth of his father's age, and in five years his age will be one third of his father's age. Find the father's present age.

13. A is 10 years older than B, and $\frac{1}{6}$ of B's age is equal to $\frac{1}{8}$ of A's age. Find their ages.

14. A is 40 years old, and B is half as old as A. In how many years will B be $\frac{6}{11}$ as old as A?

15. The sum of the ages of a father and his son is 36, and in 10 years the son's age will be $\frac{2}{3}$ of the father's age. Find their ages.

16. A post is a third of its length in the ground, a fourth of its length in water, and 14 feet above water. What is the length of the post?

17. A man left \$41,000 to his wife, two sons, and two daughters. Each son received $\frac{4}{5}$, and each daughter $\frac{3}{4}$, of the wife's share. What was the share of each?

18. A man left $\frac{1}{3}$ of his property to his wife, $\frac{1}{4}$ to his daughter, and the remainder, which was \$14,000, to his son. How much money did the man leave?

19. A man lost $\frac{2}{3}$ of his fortune and \$1000, and found that he had left $\frac{1}{3}$ of his original fortune. How much money had he at first?

20. A and B together have \$500. If A gains \$150, and B loses \$100, B's money will be $\frac{6}{5}$ of A's money. How much has each?

21. After expending one third of his money and \$20, a man had left one fourth of his money and \$30. How much money had he at first?

22. A train traveling 30 miles per hour starts $\frac{3}{4}$ of an hour before a second train that travels 35 miles an hour. In how many hours will the first train be overtaken by the second?

23. An express train starts from a certain station three hours after an accommodation train, and after traveling 160 miles overtakes the accommodation train. If the rate of the express train is $\frac{3}{5}$ of the rate of the accommodation train, what is the rate of the latter? (§ 170, Ex. 3.)

24. The length of a rectangle exceeds the width by $5\frac{1}{2}$ feet. If the length were decreased by 3 feet and the width increased by $1\frac{1}{2}$ feet, the area would not be altered. Find the dimensions.

25. If each side of a square were increased by $1\frac{1}{2}$ feet, the area would be increased by $21\frac{3}{4}$ square yards. Find the side of the square.

26. The width of a room is $\frac{4}{5}$ of its length. Another room is 10 feet shorter and 1 foot wider. If the second room contains 150 square feet less than the first room, find the dimensions of the room.

27. If the radius of a circle were increased by one yard, its area would be increased by 66 square yards. If π is assumed as $3\frac{1}{7}$, find the radius of the circle. (The area s of a circle whose radius is R , is found by the formula $s = \pi R^2$.)

28. When between one and two o'clock are the hands of a watch together? (§ 170, Ex. 1.)

X 29. At what time between 2 and 3 o'clock are the hands of a clock together?

30. At what time between 3 and 4 o'clock are the hands of a clock in a straight line and opposite?

X 31. At what time between 6 and 7 o'clock are the hands for the first time at right angles?

X 32. At what time between 6 and 7 o'clock are the hands for the second time at right angles?

33. A has invested capital at $4\frac{1}{2}\%$ and B has invested \$10,000 more than A at 4%. They both derive the same income from their investments. How much money has each invested?

X 34. A man has $\frac{1}{2}$ of his money invested at 4%, one third at $3\frac{1}{2}\%$, and the remainder at 5%. How much money has he invested if his annual interest therefrom is \$12,000?

X 35. A man has a certain sum of money invested at 5%, and twice that amount at 4%. The sum invested at 4% brings \$350 more interest than the one invested at 5%. How much money is invested at 4%?

36. What principal invested at 4% simple interest for two years will amount to \$27,000?

37. Divide 65 into two parts, such that, if the greater is divided by the smaller, the quotient is $\frac{3}{2}$.

38. The sum of two numbers is 123, and if the greater is divided by the smaller, the quotient is 2 and the remainder is 3. Find the numbers.

X 39. An ounce of gold when weighed in water loses $\frac{1}{19}$ of an ounce, and an ounce of silver $\frac{2}{21}$ of an ounce. How many ounces of gold and silver are there in a mixed mass weighing 100 ounces in air, and losing 7 ounces when weighed in water?

40. A can do a piece of work in 2 days, and B in 4 days. In how many days can both do it working together? (§ 170, Ex. 2.)

41. A can do a piece of work in 3 days, and B in 4 days. In how many days can both do it working together?

42. A can do a piece of work in 4 days, and B in 5 days. In how many days can both do it working together?

171. The last three questions and their solutions differ only in the numerical values of the two given numbers. Hence, by taking for these numerical values two general algebraic numbers, *e.g.* m and n , it is possible to solve all examples of this type by one example. Answers to numerical questions of this kind may then be found by numerical substitution. The problem to be solved, therefore, is:

A can do a piece of work in m days and B in n days. In how many days can both do it working together?

If we let x = the required number of days, and apply the method of § 170, Ex. 2, we obtain the equation $\frac{1}{m} + \frac{1}{n} = \frac{1}{x}$.

Solving, $x = \frac{mn}{m+n}$.

Therefore both working together can do it in $\frac{mn}{m+n}$ days.

To find the numerical answer, if A can do this work in 6 days and B in 3 days, make $m = 6$ and $n = 3$. Then $\frac{6 \cdot 3}{6 + 3} = 2$, *i.e.* they can both do it in 2 days.

Solve the following problems:

43. In how many days can A and B working together do a piece of work if each alone can do it in the following number of days:

- (a) A in 6, B in 6.
- (b) A in 12, B in 4.
- (c) A in 12, B in 6.
- (d) A in 20, B in 5.

44. Find three consecutive numbers whose sum equals 21.

45. Find three consecutive numbers whose sum equals 135.

The last two examples are special cases of the following problem :

46. Find three consecutive numbers whose sum equals m . Find the numbers if $m = 24$; 30,009; 918,414.

47. Find two consecutive numbers the difference of whose squares is 17.

48. If the sides of a square were increased 1 yard, the area would be increased by 23 square yards. Find the side of the square.

49. A battalion of soldiers is drawn up in the form of a solid square. To form a solid square containing in each side one soldier more, 27 more men are required. How many men are there in the square?

The last three examples are special cases of the following one :

50. The difference of the squares of two consecutive numbers is m ; find the smaller number. By using the result of this problem, solve the following ones:

I. Find two consecutive numbers, the difference of whose squares is (a) 47, (b) 121, (c) 1517, (d) 10,475,429.

II. If the sides of a square were increased by one foot, the area would be increased (a) 17, (b) 917, (c) 415,673 square feet. Find the side of the square.

51. The sum of the three angles of any triangle is equal to 180° . In a given triangle the second angle equals three times the first, and the third angle 5 times the first. Find the first angle.

52. The second angle of a triangle equals m times the first, and the third equals n times the first. Find the first angle.

Solve the problem if:

(a) the second angle = 5 times the first, the third angle = 3 times the first.

(b) the second angle = 2 times the first, the third angle = 9 times the first.

(c) the second angle = $7\frac{3}{4}$ times the first, the third angle = $6\frac{1}{4}$ times the first.

53. If the radius of a circle is increased by 1 foot, the area is increased by m square feet. Find the radius of the circle.

Solve the problem if, *by increasing* the radius by 1 foot, the area is increased (a) by 44, (b) 11, (c) 286 square feet.

(Assume $\pi = \frac{22}{7}$.)

54. Two men start at the same hour from two towns, 100 miles apart, the first traveling $3\frac{1}{2}$ miles per hour, and the second $6\frac{1}{2}$ miles per hour. After how many hours do they meet, and how many miles does each travel?

55. Two men start at the same time from two towns, d miles apart, the first traveling at the rate of m , the second at the rate of n miles per hour. After how many hours do they meet, and how many miles does each travel?

Solve the problem if the distance, the rate of the first, and the rate of the second, are respectively :

(a) 48 miles, 2 miles per hour, 4 miles per hour.

(b) 25 miles, 3 miles per hour, 7 miles per hour.

(c) 18 miles, $2\frac{1}{2}$ miles per hour, $3\frac{1}{2}$ miles per hour.

56. In how many minutes will the minute hand of a clock move over n minute spaces more than the hour hand?

NOTE. The greater number of examples relating to movements of the hands of a clock can be solved by determining from the data of the example the number of minute spaces the minute has to move more than the hour hand, and by substituting this number for n in the answer of the last example.

57. When are the hands of a clock for the first time together after (a) 1 o'clock, (b) 6 o'clock, (c) 9 o'clock?

58. When are the hands of a clock opposite, and in the same straight line after (a) 2 o'clock, (b) 4 o'clock, (c) 8 o'clock?

59. A cistern can be filled by two pipes in m and n minutes respectively. In how many minutes can it be filled by the two pipes together? Find the numerical answer, if m and n are respectively (a) 4 and 5 minutes, (b) 7 and 42 minutes, (c) 2 and 3 hours.

60. A cistern can be filled by three pipes in 3, 4, and 5 hours respectively. In how many hours can it be filled by all three together?

61. State and solve a literal problem which comprises all examples of the type of the preceding one. Find the answer if the pipes fill the cistern respectively in (a) 2, 3, 4 hours, (b) 6, 30, and 20 minutes.

62. A cistern can be filled by two pipes in 3 and 12 hours respectively, and emptied by a third pipe in 5 hours. In how many hours can the cistern be filled if all the pipes are running together? Can the last answer be obtained from the general answer of the preceding one?

63. A cistern has 3 pipes. The first will fill the cistern in 7 minutes, the second in 42 minutes, and the three running together in 5 minutes. In how many minutes will the last one fill it?

CHAPTER X

RATIO AND PROPORTION

RATIO

172. The **ratio** of two numbers is the quotient obtained by dividing the first number by the second.

Thus the ratio of a and b is $\frac{a}{b}$ or $a \div b$. The ratio is also frequently written $a : b$, the symbol $:$ being a sign of division. (In most European countries this symbol is employed as the usual sign of division.) The ratio of $12 : 3$ equals 4, $6 : 12 = .5$, etc.

173. A ratio is used to compare the magnitude of two numbers.

Thus, instead of writing " a is 5 times as large as b ," we may write $a : b = 5$.

174. The first term of a ratio is the **antecedent**, the second term the **consequent**.

In the ratio $a : b$, a is the antecedent, b is the consequent. The numerator of any fraction is the antecedent, the denominator the consequent.

175. The ratio $\frac{b}{a}$ is the **inverse** of the ratio $\frac{a}{b}$.

176. *Since a ratio is a fraction, all principles relating to fractions may be applied to ratios. E.g. a ratio is not changed if its terms are multiplied or divided by the same number, etc.*

Ex. 1. Simplify the ratio $2\frac{1}{2} : 3\frac{1}{3}$.

$$2\frac{1}{2} : 3\frac{1}{3} = \frac{5}{2} : \frac{10}{3} = \frac{5}{2} \times \frac{3}{10} = \frac{3}{4} = 3 : 4.$$

A somewhat shorter way would be to multiply each term by 6.

Ex. 2. Transform the ratio $5 : 3\frac{3}{4}$ so that the first term will equal 1.

$$5 : 3\frac{3}{4} = \frac{5}{5} : \frac{3\frac{3}{4}}{5} = 1 : \frac{3}{4}.$$

Ex. 3. Solve the equation $a : x = 9$.

$$\frac{a}{x} = 9.$$

Clearing,

$$a = 9x.$$

Dividing,

$$x = \frac{a}{9}.$$

EXERCISE 71

Find the value of the following ratios:

1. $144 : 36$.

5. $\$6.00 : \2.00 .

2. $5\frac{5}{8} : 6\frac{2}{3}$.

6. $1\frac{2}{3}$ hours : 36 seconds.

3. $\frac{1}{2} : \frac{1}{4}$.

7. 6 yards : 12 feet.

4. $16\frac{2}{3} : 1\frac{2}{3}$.

8. 6 dollars : 75 cents.

Simplify the following ratios:

9. $2\frac{1}{2} : 3\frac{1}{2}$.

14. $49x^2y : 63xy^2$.

10. $18\frac{8}{9} : 26\frac{2}{3}$.

15. $4(a+b)x : 6(a+b)y$.

11. $4\frac{1}{2} : 3$.

16. $(x^2 - 1) : (x + 1)^2$.

12. $24a : 8a$.

17. $(x^2 - 3x + 2) : (x^2 - 4x + 4)$.

13. $17xy : 8\frac{1}{2}xy$.

18. $\frac{x+1}{x-1} : \frac{x^2+1}{x^2-1}$.

Transform the following ratios so that the antecedents equal one:

19. $3a^2 : 15a^2$.

21. $4 : 12$.

20. $3\frac{3}{4} : 4\frac{2}{7}$.

22. $5 : 12$.

Solve the following equations:

23. $26 : x = 13$.

25. $a : x = b$.

24. $45 : 4x = 15$.

26. $63 : 7x = 9$.

27. $5abc : 4x = 15bc.$

30. $b : x = a - c.$

28. $16 : (x - 3) = 4.$

31. $150 : (3x + 1) = 11.$

29. $144 : (2x + 1) = 16.$

32. If 150 ounces of gold cost \$3060, and 20 ounces of silver \$12, find the ratio of the value of gold to the value of silver.

PROPORTION

177. A **proportion** is a statement expressing the equality of two ratios.

$\frac{3}{4} = \frac{6}{8}$ or $a : b = c : d$ are proportions.

178. The first and fourth terms of a proportion are the **extremes**, the second and third terms are the **means**. The last term is the **fourth proportional** to the first three.

In the proportion $a : b = c : d$, a and d are the extremes, b and c the means. The last term d is the fourth proportional to a , b , and c .

179. If the means of a proportion are equal, either mean is the **mean proportional** between the first and the last terms, and the last term the **third proportional** to the first and second terms.

In the proportion $a : b = b : c$, b is the mean proportional between a and c , and c is the third proportional to a and b .

180. Quantities of one kind are said to be *directly proportional* to quantities of another kind, if the ratio of any two of the first kind is equal to the ratio of the corresponding two of the other kind.

If 4 cm. of iron weigh 30 grams, then 6 cm. of iron weigh 45 grams, or 4 cm. : 6 cm. = 30 grams : 45 grams. Hence the weight of a mass of iron is proportional to its volume.

NOTE. Instead of "directly proportional" we may say briefly "proportional."

Quantities of one kind are said to be *inversely proportional* to quantities of another kind, if the ratio of any two of the first kind is equal to the inverse ratio of the corresponding two of the other kind.

If 6 men can do a piece of work in 4 days, then 8 men can do it in 3 days, or $6 : 8$ equals the inverse ratio of $4 : 3$, *i.e.* $3 : 4$. Hence the number of men required to do some work, and the time necessary to do it, are inversely proportional.

181. *In any proportion the product of the means is equal to the product of the extremes.*

Let $a : b = c : d$,

or $\frac{a}{b} = \frac{c}{d}$.

Clearing of fractions, $ad = bc$.

182. *The mean proportional between two numbers is equal to the square root of their product.*

Let the proportion be $a : b = b : c$.

Then $b^2 = ac$. (§ 181.)

Hence $b = \sqrt{ac}$.

183. *If the product of two numbers is equal to the product of two other numbers, either pair may be made the means, and the other pair the extremes, of a proportion. (Converse of § 181.)*

If $mn = pq$, and we divide both members by nq , we have

$$\frac{m}{q} = \frac{p}{n}.$$

Ex. 1. Find x , if $6 : x = 12 : 7$.

$$12x = 42. \quad (\S 181.)$$

Hence $x = \frac{42}{12} = 3\frac{1}{2}$.

Ex. 2. Determine whether the following proportion is correct or not:

$$8 : 5 = 7 : 4\frac{3}{8}.$$

$8 \times 4\frac{3}{8} = 35$, and $5 \times 7 = 35$; hence the proportion is correct.

184. If $a : b = c : d$, then

I. $b : a = d : c$. (Frequently called **Inversion**.)

II. $a : c = b : d$. (Called **Alternation**.)

III. $a + b : b = c + d : d$. (**Composition**.)

Or $a + b : a = c + d : c$.

IV. $a - b : b = c - d : d$. (**Division**.)

V. $a + b : a - b = c + d : c - d$. (**Composition and Division**.)

Any of these transformations may be proved by the method of § 169, although in many cases shorter proofs exist.

To prove, e.g. $\frac{a-b}{b} = \frac{c-d}{d}$.

This is true if $ad - bd = bc - bd$.

Or if $ad = bc$.

But $ad = bc$. (§ 181.)

Hence $\frac{a-b}{b} = \frac{c-d}{d}$.

185. These transformations are used to simplify proportions.

I. Change the proportion $4 : 5 = x : 6$ so that x becomes the last term.

By inversion $5 : 4 = 6 : x$.

II. Alternation shows that a proportion is not altered when its antecedents or its consequents are multiplied or divided by the same number.

E.g. to simplify $48 : 21 = 32 : 7x$, divide the antecedents by 16, the consequents by 7, $3 : 3 = 2 : x$.

Or $1 : 1 = 2 : x$, i.e. $x = 2$.

III. To simplify the proportion $5 : 6 = 4 - x : x$.

Apply composition, $11 : 6 = 4 : x$.

IV. To simplify the proportion $8 : 3 = 5 + x : x$.

Apply division, $5 : 3 = 5 : x$.

Divide the antecedents by 5, $1 : 3 = 1 : x$.

V. To simplify $\frac{m+3n}{m-3n} = \frac{m+x}{m-x}$.

Apply composition and division, $\frac{2m}{6n} = \frac{2m}{2x}$.

Or $\frac{m}{3n} = \frac{m}{x}$.

Dividing the antecedents by m , $\frac{1}{3n} = \frac{1}{x}$.

NOTE. A parenthesis is understood about each term of a proportion.

EXERCISE 72

Determine whether the following proportions are true or not:

1. $6\frac{1}{3} : 2\frac{1}{2} = 8 : 3$.

4. $7 : 10 = 5 : 7\frac{1}{7}$.

2. $10 : 5\frac{1}{2} = 4 : 2\frac{1}{5}$.

5. $8a : 5b^2 = 5a : 3\frac{1}{8}b^2$.

3. $9 : 25 = 4 : 11\frac{1}{9}$.

Simplify the following proportions and determine whether they are true or not:

6. $240 : 81 = 95 : 9$.

8. $.65 : .91 = .4 : .56$.

7. $57 : 69 = 38 : 46$.

9. $13\frac{1}{3} : 11\frac{1}{5} = 6\frac{3}{7} : 5\frac{2}{5}$.

10. $11\frac{1}{4} : 12\frac{6}{7} = 1\frac{5}{6} : 2\frac{2}{21}$.

11. $(a^2 - b^2) : (a + b)^2 = (a + b) : (a - b)$.

12. $x^3 - y^3 : x^2 + xy + y^2 = x^2 - y^2 : x + y$.

Simplify the following proportions and determine the value of x :

13. $12 : 16 = 15 : x$.

18. $96 : 72 = 4x : 21$.

14. $28 : 35 = 16 : x$.

19. $135 : 5x = 72 : 64$.

15. $25 : 55 = 35 : x$.

20. $.36 : .8x = .105 : .63$.

16. $48 : 75 = x : 32$.

21. $6\frac{2}{3} : 26\frac{1}{4} = x : 2\frac{1}{4}$.

17. $x : 38 = 15 : 19$.

22. $\frac{7}{4}ab : \frac{5}{6}bc = x : \frac{1}{8}cd$.

$$23. (a+b):(a-b)=x:\left(\frac{1}{b}-\frac{1}{a}\right).$$

$$24. \frac{25b^2}{12a^2}:\frac{5b^4}{16ac^2}=x:\frac{10a^3b^3}{3c^4}.$$

Find the fourth proportional to:

$$25. 1, 2, 3.$$

$$27. 2, 4\frac{1}{2}, 9\frac{1}{3}.$$

$$29. a^2, ab, ac.$$

$$26. 2, 3, 4.$$

$$28. a, b, c.$$

$$30. m^3, mn, mn^2.$$

Find the third proportional to:

$$31. 9 \text{ and } 6.$$

$$33. 25 \text{ and } 15.$$

$$35. 1 \text{ and } m.$$

$$32. 16 \text{ and } 20.$$

$$34. a^3b^3 \text{ and } a^2b.$$

$$36. m \text{ and } 1.$$

Find the mean proportional to:

$$37. 27 \text{ and } 3.$$

$$40. a+b \text{ and } a-b.$$

$$38. 2 \text{ and } 18.$$

$$41. a \text{ and } 4a.$$

$$39. \frac{ab}{c} \text{ and } \frac{ac}{b}.$$

$$42. 9m^2n \text{ and } 11\frac{1}{9}mn^2.$$

43. Form two proportions commencing with 3 from the equation $3 \times 6 = 2 \times 9$.

44. If $ab = xy$, form all possible proportions whose terms are a, b, x , and y .

45. If $4x = 5y$, find the ratio of $x:y$.

46. If $11x = 6y$, find the ratio of $x:y$.

47. If $ax = by$, find the ratio of $x:y$.

Transform the following equations so that x will be the fourth term:

$$48. 3:5=x:6.$$

$$50. x:2=4:3.$$

$$49. 3:x=5:9.$$

$$51. a:b=x:c.$$

Transform the following proportions so that only one term contains x :

52. $5:6=11-x:x$.

59. $6:5=7+x:x$.

53. $6:13=38-x:x$.

60. $23:21=4+x:x$.

54. $6a:5a=22-x:x$.

61. $9a:5a=20+x:x$.

55. $3:2=x:15-x$.

62. $4:7=x:x+15$.

56. $4x:5x=x:18-x$.

63. $3:2=4+x:4-x$.

57. $a:b=c-x:x$.

64. $m:n=9+x:9-x$.

58. $3:4=x-4:4$.

65. $a+b:a-b=a+x:a-x$.

Find the ratio $x:y$, if:

66. $6x=7y$.

68. $x:2=y:3$.

70. $x+y:y=7:2$.

67. $y:x=7:2$.

69. $4:y=8:x$.

71. $x-y:y=2:3$.

72. $x+y:x-y=3:1$.

75. If $\frac{x}{y}=\frac{3}{2}$, find $\frac{x+y}{y}$.

73. $3x+4y=5x+2y$.

74. $\frac{4x-3y}{2x+5y}=\frac{3}{8}$.

76. If $\frac{x}{y}=\frac{12}{7}$, find $\frac{x+y}{x-y}$.

77. State whether the quantities mentioned below are directly or inversely proportional:

(a) The number of yards of a certain kind of silk, and their cost.

(b) The time a train needs to travel 10 miles, and the speed of the train.

(c) The length of a rectangle of constant width, and the area of the rectangle.

(d) The sum of money producing \$60 interest at 5%, and the time necessary for it.

(e) The distance traveled by a train moving at a uniform rate, and the time.

78. A line 4 inches long on a certain map corresponds to 42 miles. A line 5 inches long corresponds to how many miles?

79. The areas of two circles are proportional to the squares of their radii. The radii of two circles are to each other as

3:5, and the area of the smaller circle is 18 square inches. Find the area of the larger.

80. The temperature remaining the same, the volume of a gas is inversely proportional to the pressure. A body of gas under a pressure of 15 pounds per square inch has a volume of 2 cubic feet. What will be the volume if the pressure is 6 pounds per square inch?

81. Two rectangles have equal areas, and the ratio of their longer sides is 3:19. Find the ratio of their shorter sides.

82. The distance one can see from an elevation of h feet is very nearly the mean proportional between the elevation h and the diameter of the earth (8000 miles), provided these lengths are expressed in the same denomination. What is the greatest distance a person can see from a ship 30 feet high? From the Eiffel Tower (900 feet high)? From Mount Etna (10,000 feet high)?

186. *When a problem requires the finding of two numbers which are to each other as $m:n$, it is advisable to represent these unknown numbers by mx and nx .*

Ex. 1. Divide 108 into two parts which are to each other 11:7.

This problem contains two statements:

I. The sum of two numbers is 108.

II. The ratio of the same numbers is 11:7.

A. If we use I to express one unknown number by the other, one number is x , the other $108 - x$, and II, written in symbols, produces $x:108 - x = 11:7$, which solved gives the value of x .

B. It is better, however, to use II to express the two numbers.

Let	$11x =$ the first number,
then	$7x =$ the second number.
I, expressed in symbols, gives	$11x + 7x = 108,$
or	$18x = 108.$
Therefore	$x = 6.$
Hence	$11x = 66$ is the first number,
and	$7x = 42$ is the second number.

Ex. 2. A line AB , 4 inches long, is produced to a point C , so that $(AC):(BC)=7:5$. Find AC and BC .



Let	$AC = 7x$.
Then	$BC = 5x$.
Hence	$AB = 2x$.
Or	$2x = 4$
	$x = 2$.
Therefore	$7x = 14 = AC$.
	$5x = 10 = BC$.

EXERCISE 73

1. Divide 120 in the ratio of 7:8.
2. Divide 35 in the ratio of 11:3.
3. Two numbers are to each other as 9:7, and their difference is 14. Find the numbers.
4. A straight line 16 inches long is divided in the ratio 3:5. What are the parts?
5. Brass is an alloy consisting of two parts of copper and 1 part of zinc. How many ounces of copper and of zinc are there in 8 ounces of brass?
6. Gunmetal consists of 9 parts of copper and one part of tin. How many ounces of each are there in 17 ounces of gunmetal?
7. Air is a mixture composed mainly of oxygen and nitrogen, whose volumes are to each other as 21:79. How many cubic feet of oxygen are there in a room whose volume is 3550 cubic feet?
8. A line AB , 9 inches long, is produced to a point C , and AC is to BC as 13:11. Find the length of AC and BC .
9. The total area of land is to the total area of water as 7:18. If the total surface of the earth is 197,000,000 square miles, find the number of square miles of land and of water.

10. Water consists of one part of hydrogen and 8 parts of oxygen. How many grams of hydrogen are contained in 100 grams of water?

11. Divide 100 in the ratio of $a : b$.

12. Divide a in the ratio of $b : 1$.

13. The three sides of a triangle are respectively 9, 18, and 21 inches, and the longest side is divided in the ratio of the other two. How long are the parts?

14. A line 10 inches long is divided in the ratio of $a : 2$. Find the parts.

15. The three sides of a triangle are respectively a , b , and c inches. If c is divided in the ratio of the other two, what are its segments?

16. A's age is to B's age as 3 : 2, and 5 years ago the sum of their ages was 45 years. Find their ages.

17. A's age is to B's age as 4 : 3, and 8 years ago the ratio of their ages was 3 : 2. Find their ages.

18. Two men, A and B, start from the same town and travel in the same direction. A's rate of travel is to B's rate as 5 : 2. Find their rates of travel, if after three hours they are 9 miles apart.

19. One angle of a triangle is 67° and the ratio of the two others is 3 : 5. How many degrees are contained in these angles?

187. *The products of the corresponding terms of two or more proportions are in proportion.*

If $\frac{a}{b} = \frac{c}{d}$ and $\frac{m}{n} = \frac{p}{q}$, then $\frac{am}{bn} = \frac{cp}{dq}$.

This follows directly from § 154.

Ex. 1. If $a : b = 3 : 4$, and $b : c = 7 : 5$, find the ratio of $a : c$.

$$ab : bc = 21 : 20. \quad (\S 187.)$$

Or simplifying,

$$a : c = 21 : 20.$$

188. If

$$a : b = 3 : 4,$$

and

$$b : c = 4 : 5,$$

then

$$a : c = 3 : 5. \quad (\S 187.)$$

These three proportions can be conveniently written as follows:

$$a : b : c = 3 : 4 : 5.$$

Ex. 2. Divide 100 in the ratio $a : b : c$.

Let

$$ax = \text{the first part.}$$

Then

$$bx = \text{the second part,}$$

and

$$cx = \text{the third part.}$$

Hence

$$ax + bx + cx = 100.$$

Uniting,

$$x(a + b + c) = 100,$$

$$x = \frac{100}{a + b + c}.$$

Therefore

$$ax = \frac{100a}{a + b + c}, \text{ the first part.}$$

$$bx = \frac{100b}{a + b + c}, \text{ the second part.}$$

$$cx = \frac{100c}{a + b + c}, \text{ the third part.}$$

Ex. 3. If $\frac{x}{y} = 5$, find $\frac{2x - 7y}{3x - 14y}$.

Dividing each term by y ,

$$\frac{2x - 7y}{3x - 14y} = \frac{2\left(\frac{x}{y}\right) - 7}{3\left(\frac{x}{y}\right) - 14} = \frac{10 - 7}{15 - 14} = \frac{3}{1} = 3.$$

Ex. 4. If $a : b = c : d$, prove that $\frac{ab + cd}{ab - cd} = \frac{a^2 + c^2}{a^2 - c^2}$.

According to § 169, this is true if

$$a^3b + a^2cd - abc^2 - c^3d = a^3b + abc^2 - a^2cd - c^3d.$$

Or if $a^2cd - abc^2 = abc^2 - a^2cd$. (Canceling.)

Or if $ad - bc = bc - ad$. (Dividing by ac .)

Or if $2ad = 2bc$. (Transposing.)

Or if $ad = bc$. (Dividing by 2.)

But since the $ad = bc$, $\frac{ab + cd}{ac - cd} = \frac{a^2 + c^2}{a^2 - c^2}$.

NOTE. The methods of simplifying proportions, as given in § 184, will frequently produce shorter proofs than the method used in Ex. 4.

EXERCISE 74

1. If $a : b = 3 : 4$, and $b : c = 5 : 6$, find $a : c$.
2. If $a : b = 3 : x$, and $b : c = x : 5$, find $a : c$.
3. If $P : Q = a : b$, and $Q : R = c : d$, find the ratio $P : R$ in terms of a, b, c , and d .
4. If $a : b = 2 : 3$, and $b : c = 2 : 9$, find $a : c$.
5. If $a : b = m : n$, and $b : c = m : n$, find $a : c$ in terms of m and n .
6. If $a : b = 2 : 1$, $b : c = 2 : 3$, and $c : d = 4 : 5$, find $a : d$.
7. If $a : b = b : c = 1 : 2$, find $a : c$.
8. If $a : b = b : c = m : n$, find $a : c$ in terms of m and n .
9. If $x^2 : \frac{1}{y} = 3 : 4$, and $\frac{1}{x} : y^2 = 8 : 15$, find $x : y$.
10. Divide 143 into three parts which are to each other as $2 : 4 : 5$.
11. The three angles of a triangle are to each other as $6 : 11 : 19$. Find the number of degrees contained in each if the sum of the 3 angles is 180° .

12. A line 12 inches long is divided into 3 parts which are to each other as 6 : 7 : 9. How long is each part?

13. A line a inches long is divided into 3 parts which are to each other as 3 : 4 : 5. Find the parts.

14. The sum of three sides (perimeter) of a triangle is 90 inches, and the ratio of the 3 sides equals 3 : 4 : 5. Find each side.

15. The perimeter of a triangle equals p inches, and the ratio of the sides is $a : b : c$. Find each side.

16. If $x : y = 5 : 1$, find $2x + 3y : 3x - 2y$.

17. If $a : b = 4 : 3$, find $6a - b : 9a - 7b$.

18. If $m : n = 5 : 2$, find $8m - 7n : 4m - 13n$.

19. If $a : b = -3$, find $6a - 2b : 9a + 17b$.

If $a : b = c : d$, prove that:

$$20. ac : bd = c^2 : d^2.$$

$$21. a + b : a + b + c + d = a : a + c.$$

$$22. a - c : b - d = c : d.$$

$$23. a^2 + c^2 : b^2 + d^2 = a^2 : b^2.$$

$$24. a^2 + b^2 : c^2 + d^2 = a^2 : c^2.$$

$$25. a^2 + 2b^2 : 2b^2 = ac + 2bd : 2bd.$$

$$26. a^2 + 3b^2 : a^2 - 3b^2 = c^2 + 3d^2 : c^2 - 3d^2.$$

$$27. a^2 + b^2 : ac + bd = b : d.$$

$$28. a^2 + 2b^2 : ac + 2bd = a : c.$$

If $a : b = b : c$, prove that

$$29. a^2 : b^2 = ab : bc.$$

$$30. a^2 + ab : b^2 + bc = a^2 : b^2.$$

$$31. a - b : b - c = b : c.$$

$$32. \text{ If } a^2 + b^2 + c^2 + d^2 : b^2 + d^2 = c^2 + d^2 : d^2, \text{ prove that } a : b = c : d$$

Solve the following equations:

$$33. \quad 8x - 7 : 6x + 7 = 12x - 19 : 9x + 2.$$

$$34. \quad 5x - 24 : 3x + 32 = 7x + 6 : 9x - 8.$$

$$35. \quad 12x^2 - 7x + 5 : 18x^2 - 11x + 9 = 2 : 3.$$

$$36. \quad 8x^2 - 3x + 4 : 12x^2 + 5x - 13 = 2 : 3.$$

$$37. \quad x - a : x - b = a + b : a - b.$$

HINT. Simplify before solving.

$$38. \quad x : a - b = x - b : a.$$

$$39. \quad a^2 - b^2 : ax - b = a + b : x.$$

$$40. \quad x - a + b : x + a - b = a - x + 2b : b - x + 2a.$$

41. Two travelers starting at two points 24 miles distant walk towards each other, and their rates are as 4:5. How many miles does each walk before they meet?

42. The three sides of a triangle are to each other as 3:4:5, and the area is 24 square inches. How long is each side? (Compare § 30.)

REVIEW EXERCISE III

Solve the equations:

$$1. \quad \frac{1 - \frac{1}{x}}{1 + \frac{1}{x}} + \frac{1 + \frac{1}{x+2}}{1 - \frac{1}{x+2}} = 2 + x.$$

$$2. \quad \frac{x}{x+1} + \frac{2x}{2x+3} - \frac{2x}{x+2} = \frac{3x^2+4}{(x+1)(2x+3)(x+2)}.$$

$$3. \quad a^3 + b^3 : x = a^2 - ab + b^2 : a + b.$$

$$4. \quad 1 : \frac{1}{2x-5} = \frac{1}{4} : \frac{1}{3x-5}.$$

5. The sum of two numbers is 79, and if the greater is divided by the smaller, the quotient is $\frac{4}{3}$, and the remainder is 2. Find the numbers.

6. A boy bought some oranges at 3ϕ apiece. He sold $\frac{4}{5}$ of them for 4ϕ apiece, and the rest for 5ϕ apiece, and gained 24ϕ . How many oranges did he buy?

7. A man sold a house for \$4000 more than half its cost, and gained thereby \$1000. What did he pay for the house?

8. A man sold a house for \$2000 more than $\frac{3}{4}$ of the cost, and gained thereby 25% on the money invested. What did he pay for the house?

9. Simplify $\frac{\frac{m+n}{m-n} - 1}{\frac{m-n}{m+n} + 1} \div \frac{m+n}{m-n}$.

10. Simplify $\left[\frac{a^4+1}{a^2+1} - a^4 \right] \div \frac{1-a^6}{1+a^2}$, and check the answer by the numerical substitution, $a = 2$.

11. Simplify $\left\{ 1 + \frac{\frac{x-y}{x+y} - \frac{x+y}{x-y}}{\frac{x-y}{x+y} + \frac{x+y}{x-y}} \right\} \div \left\{ 1 - \frac{\frac{x+y}{x-y} + \frac{x-y}{x+y}}{\frac{x+y}{x-y} - \frac{x-y}{x+y}} \right\}$.

12. Simplify $(5x^{3m+2}y^{4p} - 7y^{4m+5}z^{5q})(5x^{3m+2}y^{4p} + 7y^{4m+5}z^{5q})$.

13. Reduce to lowest terms:

$$\frac{(3abc + 4bcd - 5cde)(36p^2 - 25q^2)}{(6p + 5q)(6abc^2 - 10c^2de + 8bc^2d)}$$

14. Find two numbers whose sum equals a and whose difference equals b .

15. Divide 300 into two parts such that $\frac{1}{5}$ of the greater increased by twice the smaller is as much as the smaller increased by $\frac{2}{5}$ of the greater.

16. Factor I. $x^4 - (a+4)x^2 + 4a$.

II. $9(2a-x)^2 - 4(3a-x)^2$.

17. If $a:b = 3:2$, and $b:c = 2:7$, find $a:c$.

18. Solve $\frac{c-a}{b(c-x)} = \frac{b-a}{b(b-x)}$.

19. Find the L. C. M. of:

$(9x^2 - 16y^2)^2, 9x^2 - 16y^2, 3x + 4y, 3x - 4y.$

20. If $a = 5$, find the numerical value of:

$$.7a^5 - .5a^4 - .9a^5 + .7a^4 + .1a^5 - 1.2a^4.$$

21. Find the H. C. F. of:

$$a^{3m} - 1, a^{2m} - 1, (a^m - 1)^4.$$

22. Find the H. C. F. of:

$$x^{3m-3} + y^{3n-3}, \text{ and } x^{2m-2} - y^{2n-2}.$$

23. Reduce to mixed or integral expressions:

$$\frac{a^4 + b^4}{a + b}.$$

24. Simplify $\left(\frac{a^{3n+4}b^{4x-1}}{c^{2r-5}} \div \frac{a^{2n-5}b^{3x-4}}{c^{4r+3}} \right) \div \frac{a^{7-4n}b^{x-3}}{c^{r-5}}.$

25. Simplify:

$$\frac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^2(x-a)(x-c)}{(b-a)(b-c)} + \frac{c^2(x-a)(x-b)}{(c-a)(c-b)}.$$

26. Solve:

$$\frac{3-x}{8-x} + \frac{8-x}{6-x} + \frac{2-x}{4-x} = \frac{10-x}{8-x} - \frac{x+2}{6-x} + \frac{5-x}{4-x}.$$

CHAPTER XI

SIMULTANEOUS LINEAR EQUATIONS

189. An equation of the first degree containing two or more unknown numbers can be satisfied by any number of values of the unknown quantities.

If $2x - 3y = 5.$ (1)

Then $y = \frac{2x - 5}{3}.$ (2)

I.e. if $x = 0, y = -\frac{5}{3}.$

If $x = \frac{1}{2}, y = -\frac{4}{3}.$

If $x = 1, y = -1, \text{ etc.}$

Hence, the equation is satisfied by an infinite number of sets of values. Such an equation is called *indeterminate*.

However, if there is given another equation, expressing a different relation between x and y , such as

$$x + y = 10; \quad (3)$$

these unknown numbers can be found.

From (3) it follows $y = 10 - x$, and since the equations have to be satisfied by the same values of x and y , the two values of y must be equal.

Hence $\frac{2x - 5}{3} = 10 - x.$ (4)

The root of (4) is $x = 7$, which substituted in (2) gives $y = 3$.

Therefore, if both equations are to be satisfied by the same values of x and y , there is only one solution.

190. A system of simultaneous equations is a group of equations that can be satisfied by the same values of the unknown numbers.

$x + 2y = 5$ and $7x - 3y = 1$ are simultaneous equations, for they are satisfied by the values $x = 1, y = 2$. But $2x - y = 5$ and $4x - 2y = 6$ are not simultaneous, for they cannot be satisfied by any value of x and y . The first set of equations is also called *consistent*, the last set *inconsistent*.

191. Independent equations are equations representing different relations between the unknown quantities; such equations cannot be reduced to the same form.

$5x + 5y = 50$, and $3x + 3y = 30$ can be reduced to the same form; viz. $x + y = 10$. Hence they are not independent, for they express the same relation. Any set of values satisfying $5x + 5y = 50$ will also satisfy the equation $3x + 3y = 30$.

192. A system of two simultaneous equations containing two unknown quantities is solved by combining them so as to obtain one equation containing only one unknown quantity.

The process of combining several equations so as to make one unknown quantity disappear is called **elimination**.

193. The three methods of elimination most frequently used are:

- I. *By Addition or Subtraction.*
- II. *By Substitution.*
- III. *By Comparison.*

ELIMINATION BY ADDITION OR SUBTRACTION

$$\text{194. Ex. 1. Solve } \begin{cases} 3x + 2y = 13, & (1) \\ 2x - 7y = -8. & (2) \end{cases}$$

$$\text{Multiply (1) by 2,} \quad 6x + 4y = 26. \quad (3)$$

$$\text{Multiply (2) by 3,} \quad 6x - 21y = -24. \quad (4)$$

$$\text{Subtract (4) from (3),} \quad 25y = 50.$$

$$\text{Therefore,} \quad y = 2.$$

Substitute this value of y in either of the given equations, preferably the simpler one (1),

$$3x + 4 = 13.$$

Therefore

$$x = 3.$$

$$y = 2.$$

In general, eliminate the letter whose coefficients have the lowest common multiple.

Check.

$$3 \cdot 3 + 2 \cdot 2 = 9 + 4 = 13,$$

$$2 \cdot 3 - 7 \cdot 2 = 6 - 14 = -8.$$

$$\text{Ex. 2. Solve } \begin{cases} 5x - 3y = 47, & (1) \\ 13x + 5y = 135. & (2) \end{cases}$$

$$\text{Multiply (1) by 5,} \quad 25x - 15y = 235. \quad (3)$$

$$\text{Multiply (2) by 3,} \quad 39x + 15y = 405. \quad (4)$$

$$\text{Add (3) and (4),} \quad 64x = 640.$$

$$\text{Therefore} \quad x = 10. \quad (5)$$

$$\text{Substitute (5) in (1),} \quad 50 - 3y = 47.$$

$$\text{Transposing,} \quad -3y = -3.$$

$$\text{Therefore} \quad y = 1.$$

$$x = 10.$$

$$\text{Check. } 5 \cdot 10 - 3 \cdot 1 = 47,$$

$$13 \cdot 10 + 5 \cdot 1 = 135.$$

195. Hence to eliminate by addition or subtraction:

Multiply, if necessary, the equations by such numbers as will make the coefficients of one unknown quantity equal.

If the signs of these coefficients are like, subtract the equations; if unlike, add the equations.

EXERCISE 75

Solve the following systems of equations and check the answers:

$$1. \quad \begin{cases} 5x + 7y = 50, \\ 9x + 14y = 97. \end{cases}$$

$$2. \quad \begin{cases} 12x - 13y = 9, \\ 17y - 4x = 35. \end{cases}$$

3. $\begin{cases} 8x - 5y = 49, \\ 7x + 15y = 101. \end{cases}$
4. $\begin{cases} 10x + 3y = 23, \\ 5y - 2x = 1. \end{cases}$
5. $\begin{cases} 2x + 5y = 1, \\ 6x + 7y = 3. \end{cases}$
6. $\begin{cases} 7x + 3y = 100, \\ 3x - y = 20. \end{cases}$
7. $\begin{cases} 8x - 15y = -30, \\ 2x + 3y = 15. \end{cases}$
8. $\begin{cases} 7x - 3y = 27, \\ 5x - 6y = 0. \end{cases}$
9. $\begin{cases} 7x - 3y = 15, \\ 5x + 6y = 27. \end{cases}$
10. $\begin{cases} 3x - 4y = 11, \\ 5x - 3y = 33. \end{cases}$
11. $\begin{cases} 7x + y = 42, \\ 3x - 2y = 1. \end{cases}$
12. $\begin{cases} 3x + 4y = 43, \\ 4x + 7y = 69. \end{cases}$
13. $\begin{cases} 7x - 3y = 23, \\ 3x + 4y = 31. \end{cases}$
14. $\begin{cases} 5x - 7y = -4, \\ 9x + 11y = 40. \end{cases}$
15. $\begin{cases} x + y = 100, \\ x - y = 12. \end{cases}$
16. $\begin{cases} 9x + 14y = 83, \\ 39x - 35y = -23. \end{cases}$
17. $\begin{cases} 2x - 11y = -95, \\ x - 3y = 0. \end{cases}$
18. $\begin{cases} 3x - 30y = 15, \\ 2x + 10y = 40. \end{cases}$
19. $\begin{cases} 13x - 11y = 131, \\ 19x - 24y = 33. \end{cases}$
20. $\begin{cases} 8x + 3y = 37, \\ 8y - 3x = 50. \end{cases}$
21. $\begin{cases} x - 5y = 4, \\ 3x + 5y = 32. \end{cases}$
22. $\begin{cases} 16x - 15y = 18, \\ 2x + 5y = 16. \end{cases}$
23. $\begin{cases} 4x - 7y = -5, \\ 5x + y = \frac{7}{2}. \end{cases}$
24. $\begin{cases} \frac{1}{3}x + \frac{1}{4}y = 6, \\ 3x - 4y = 4. \end{cases}$
25. $\begin{cases} 1.5x - 2y = 1, \\ 2.5x - 3y = 6. \end{cases}$
26. $\begin{cases} \frac{3}{4}x - 2y = 1, \\ \frac{1}{3}x - y = 0. \end{cases}$
27. $\begin{cases} 4.9y - 3.2x = 1.9, \\ 3.5y - 2.4x = .9. \end{cases}$
28. $\begin{cases} 1.5x - 1.1y = .01, \\ 2x - 1.7y = .08. \end{cases}$

ELIMINATION BY SUBSTITUTION

$$196. \text{ Solve } \begin{cases} 2x - 7y = -8, \\ 3x + 2y = 13. \end{cases} \quad (1)$$

(2)

Transposing $-7y$ in (1) and dividing by 2, $x = \frac{7y-8}{2}$.

Substituting this value in (2), $3\left(\frac{7y-8}{2}\right) + 2y = 13$.

Clearing of fractions, $21y - 24 + 4y = 26$.

$$25y = 50.$$

Therefore

$$y = 2.$$

This value substituted in either (1) or (2) gives $x = 3$.

197. Hence to eliminate by substitution:

Find in one equation the value of an unknown quantity in terms of the other. Substitute this value for one unknown quantity in the other equation, and solve the resulting equation.

EXERCISE 76

Solve by substitution:

- | | |
|--|--|
| 1. $\begin{cases} 3x + 2y = 5, \\ 2x + 5y = 7. \end{cases}$ | 8. $\begin{cases} 5x + 5 = 10y, \\ 5x - 2y = 16. \end{cases}$ |
| 2. $\begin{cases} 4x - 7y = 2, \\ 12x - 25y = -2. \end{cases}$ | 9. $\begin{cases} 2x = 5y - 11, \\ 3x = 27 - 7y. \end{cases}$ |
| 3. $\begin{cases} 9y - 5x = 2, \\ 3y + 4x = 29. \end{cases}$ | 10. $\begin{cases} 4y = 3x + 4, \\ 5y = 4x + 3. \end{cases}$ |
| 4. $\begin{cases} 21x - 9y = -3, \\ 3x + 5y = 31. \end{cases}$ | 11. $\begin{cases} 11x = 8y + 7, \\ 11x = 13y - 23. \end{cases}$ |
| 5. $\begin{cases} 20y - 9x = 26, \\ 40y - 27x = -2. \end{cases}$ | 12. $\begin{cases} x + 5y = 29, \\ x + 3y = 19. \end{cases}$ |
| 6. $\begin{cases} 35x - 17y = 59, \\ 7x - 5y = -9. \end{cases}$ | 13. $\begin{cases} \frac{1}{3}x - \frac{1}{4}y = 4, \\ \frac{1}{5}x + \frac{1}{4}y = 12. \end{cases}$ |
| 7. $\begin{cases} 6y - 7x = 35, \\ 24y - 41x = -29. \end{cases}$ | 14. $\begin{cases} \frac{1}{4}x + \frac{1}{6}y = 18, \\ \frac{1}{3}x - \frac{1}{6}y = 10. \end{cases}$ |

ELIMINATION BY COMPARISON

$$198. \text{ Ex. Solve } \begin{cases} 3x - 2y = 2, & (1) \\ x - 7 = -12y. & (2) \end{cases}$$

$$\text{From (1),} \quad x = \frac{2 + 2y}{3}. \quad (3)$$

$$\text{From (2),} \quad x = 7 - 12y. \quad (4)$$

$$\text{Therefore} \quad \frac{2 + 2y}{3} = 7 - 12y.$$

$$\text{Clearing of fractions,} \quad 2 + 2y = 21 - 36y.$$

$$\text{Transposing and uniting,} \quad 38y = 19.$$

$$\text{Therefore} \quad y = \frac{1}{2}.$$

$$\text{Substitute in (4),} \quad x = 1.$$

199. Hence to eliminate by comparison:

In each equation find the value of one unknown quantity in terms of the other. Form an equation with these values and solve it.

EXERCISE 77

Solve by comparison:

$$1. \quad \begin{cases} 9y = 2x - 31, \\ 9y = 5 - 16x. \end{cases}$$

$$6. \quad \begin{cases} 10x + 7y + 4 = 0, \\ 6x + 5y + 2 = 0. \end{cases}$$

$$2. \quad \begin{cases} x + 4y = 37, \\ 2x + 5y = 53. \end{cases}$$

$$7. \quad \begin{cases} \frac{1}{3}x + \frac{1}{4}y = 6, \\ 3x - 4y = 4. \end{cases}$$

$$3. \quad \begin{cases} 7x + 3y = 100, \\ 3x - y = 20. \end{cases}$$

$$8. \quad \begin{cases} 5x - 4.9y = 1, \\ 3x - 2.9y = 1. \end{cases}$$

$$4. \quad \begin{cases} 5x + 3y + 2 = 0, \\ 3x + 2y + 1 = 0. \end{cases}$$

$$9. \quad \begin{cases} 7x - 10y = .1, \\ 11x - 16y = .1. \end{cases}$$

$$5. \quad \begin{cases} 21x + 8y + 66 = 0, \\ 23y - 28x + 13 = 0. \end{cases}$$

200. Whenever one unknown quantity can be removed without clearing of fractions, it is advantageous to do so; in most cases, however, the equation must be cleared of fractions and simplified before elimination is possible.

$$\text{Ex. Solve } \begin{cases} \frac{x+2}{3} + \frac{y+3}{4} = 3, \\ \frac{x+3}{2} - \frac{y+2}{7} = 1. \end{cases} \quad (1)$$

(2)

Multiplying (1) by 12 and (2) by 14,

$$4x + 8 + 3y + 9 = 36. \quad (3)$$

$$7x + 21 - 2y - 4 = 14. \quad (4)$$

$$\text{From (3),} \quad 4x + 3y = 19. \quad (5)$$

$$\text{From (4),} \quad 7x - 2y = -3. \quad (6)$$

Multiplying (5) by 2 and (6) by 3,

$$8x + 6y = 38. \quad (7)$$

$$21x - 6y = -9. \quad (8)$$

$$\text{Adding (7) and (8),} \quad 29x = 29.$$

$$x = 1.$$

$$\text{Substituting in (6),} \quad 7 - 2y = -3.$$

$$y = 5.$$

$$\text{Check. } \frac{1+2}{3} + \frac{5+3}{4} = 1 + 2 = 3,$$

$$\frac{1+3}{2} - \frac{5+2}{7} = 2 - 1 = 1.$$

EXERCISE 78

Solve the following systems of equations by any method, and check the answers of Exs. 1-10:

$$1. \quad \begin{cases} 2(x+3) + 3(y+4) = 26, \\ 4(x+5) + 5(y+6) = 64. \end{cases}$$

$$2. \quad \begin{cases} 6(x-7) - 7(y-8) = 18, \\ 8(x-9) - 9(y-10) = 26. \end{cases}$$

$$3. \quad \begin{cases} 2(3x-4) + 3(4y-5) = 43, \\ 4(5x-6) + 5(6y-7) = 121. \end{cases}$$

$$4. \begin{cases} 2.3x + 4.7y = 70, \\ 3.4x + 5.6y = 90. \end{cases} \quad 5. \begin{cases} 5(x+2) - 3(y+1) = 23, \\ 3(x-2) + 5(y-1) = 19. \end{cases}$$

$$6. \begin{cases} 3(2x-y) + 4(x-2y) = 87, \\ 2(3x-y) - 3(x-y) = 82. \end{cases}$$

$$7. \begin{cases} \frac{5}{x+2y} = \frac{7}{2x+y}, \\ \frac{7}{3x-2} = \frac{5}{6y-7}. \end{cases}$$

$$11. \begin{cases} \frac{3x+1}{4-2y} = \frac{4}{3}, \\ x+y=1. \end{cases}$$

$$8. \begin{cases} \frac{1}{3x+1} = \frac{2}{5y+4}, \\ \frac{1}{4x-3} = \frac{2}{7y-6}. \end{cases}$$

$$12. \begin{cases} \frac{7-2x}{5-3y} = \frac{3}{2}, \\ y-x=4. \end{cases}$$

$$9. \begin{cases} \frac{x+3y}{x-7} = 8, \\ \frac{7x-13}{3y-5} = 4. \end{cases}$$

$$13. \begin{cases} \frac{x-3}{y+2} = \frac{2}{3}, \\ \frac{x+1}{y-2} = \frac{3}{2}. \end{cases}$$

$$10. \begin{cases} \frac{15x+1}{45-y} = 8, \\ \frac{12y+19}{x-10} = 25. \end{cases}$$

$$14. \begin{cases} \frac{x+2y+1}{2x-y+1} = 2, \\ \frac{3x-y+1}{x-y+3} = 5. \end{cases}$$

$$15. \begin{cases} \frac{x+3y+13}{4x+5y-25} = 3, \\ \frac{8x+y+6}{5x+3y-23} = 5. \end{cases}$$

$$16. \begin{cases} \frac{x+1}{3} - \frac{y+2}{4} = \frac{2(x-y)}{5}, \\ \frac{x-3}{4} - \frac{y-3}{3} = 2y-x. \end{cases}$$

$$17. \begin{cases} \frac{3x-2y}{5} + \frac{5x-3y}{3} = x+1, \\ \frac{2x-3y}{3} + \frac{4x-3y}{2} = y+1. \end{cases}$$

$$18. \begin{cases} (x-4)(y+7) = (x-3)(y+4), \\ (x+5)(y-2) = (x+2)(y-1). \end{cases}$$

$$19. \begin{cases} (x+3)(y+5) = (x+1)(y+8), \\ (2x-3)(5y+7) = 2(5x-6)(y+1). \end{cases}$$

$$20. \begin{cases} \frac{2x}{15} + \frac{7y}{12} = 3, \\ \frac{7x}{25} - \frac{5y}{16} = \frac{3}{20}. \end{cases}$$

$$21. \begin{cases} \frac{3x+4y}{12} - \frac{5y-3x}{9} = \frac{2x+3y}{6} - \frac{3y-2x}{3}, \\ \frac{1}{4}(x+4) + \frac{1}{10}(y-3) = \frac{3x}{4} - \frac{y-21}{2}. \end{cases}$$

$$22. \begin{cases} x:y = 3:4, \\ (x-1):(y+2) = 1:2. \end{cases} \quad 23. \begin{cases} (x+4):(y+1) = 2:3, \\ (x+2):(y-1) = 3:1. \end{cases}$$

$$24. (x+1):(y+1):(x+y) = 3:4:5.$$

201. In many equations it is advantageous at first not to consider x and y as unknown quantities, but some expressions involving x , and y , such as $\frac{1}{x}$ and $\frac{1}{y}$, xy , etc.

$$\text{Ex. 1. Solve } \begin{cases} \frac{3}{x} + \frac{8}{y} = 3. & (1) \\ \frac{15}{x} - \frac{4}{y} = 4. & (2) \end{cases}$$

$$2 \times (2), \quad \frac{30}{x} - \frac{8}{y} = 8. \quad (3)$$

$$(1) + (3), \quad \frac{33}{x} = 11.$$

$$\text{Clearing of fractions,} \quad \begin{aligned} 33 &= 11x. \\ x &= 3. \end{aligned}$$

Substituting $x = 3$ in (1), $1 + \frac{8}{y} = 3.$

$$\frac{8}{y} = 2.$$

Therefore $y = 4.$

Check. $\frac{3}{3} + \frac{8}{4} = 1 + 2 = 3$; $\frac{15}{3} - \frac{4}{4} = 5 - 1 = 4.$

Examples of this type, however, can also be solved by the regular method, provided they do not involve more than two unknown quantities.

Clearing (1) and (2) of fractions, $3y + 8x = 3xy.$ (4)

$$15y - 4x = 4xy. \quad (5)$$

$2 \times (5),$ $30y - 8x = 8xy. \quad (6)$

$(4) + (6),$ $33y = 11xy. \quad (7)$

Dividing by $11y,$ $3 = x, \text{ etc.}$

Ex. 2. Solve $\begin{cases} xy - \frac{1}{x} = 2. & (1) \\ 6xy + \frac{3}{x} = 21. & (2) \end{cases}$

$6 \times (1),$ $6xy - \frac{6}{x} = 12. \quad (3)$

$(2) - (3),$ $\frac{9}{x} = 9.$

Therefore $x = 1.$

Substituting $x = 1$ in (1), $y - 1 = 2.$

$$y = 3.$$

EXERCISE 79

Solve:

1. $\begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{5}{6}, \\ \frac{1}{x} - \frac{1}{y} = \frac{1}{6}. \end{cases}$

3. $\begin{cases} \frac{1}{x} + \frac{2}{y} = 10, \\ \frac{3}{x} + \frac{4}{y} = 24. \end{cases}$

5. $\begin{cases} \frac{3}{x} - \frac{2}{y} = -5, \\ \frac{2}{x} - \frac{5}{y} = -7. \end{cases}$

2. $\begin{cases} \frac{12}{x} + \frac{y}{3} = 5, \\ \frac{16}{x} - \frac{y}{3} = 2. \end{cases}$

4. $\begin{cases} \frac{2}{x} + \frac{4}{y} = -5, \\ \frac{11}{x} - \frac{7}{y} = \frac{3}{2}. \end{cases}$

6. $\begin{cases} \frac{5}{x} + \frac{6}{y} = 28, \\ \frac{6}{x} - \frac{4}{y} = 0. \end{cases}$

$$\begin{array}{lll}
 7. \begin{cases} \frac{7}{x} - \frac{12}{y} = \frac{5}{6}, \\ \frac{4}{y} = \frac{9}{x} - 2\frac{1}{2}. \end{cases} & 10. \begin{cases} \frac{5}{x} + \frac{6y}{5} = 10\frac{2}{5}, \\ \frac{3}{2x} - \frac{6y}{5} = 2\frac{3}{5}. \end{cases} & 13. \begin{cases} \frac{1}{3x} + \frac{1}{4y} = 2, \\ \frac{1}{x} - \frac{1}{2y} = 1. \end{cases} \\
 8. \begin{cases} \frac{24}{x} + \frac{55}{y} = 8, \\ \frac{48}{x} = \frac{43}{y} + 2\frac{1}{11}. \end{cases} & 11. \begin{cases} x + \frac{9}{y} = 13, \\ 3x - \frac{6}{y} = 6. \end{cases} & 14. \begin{cases} \frac{2}{x} + \frac{11}{2y} = 17, \\ \frac{17}{6x} - \frac{5}{y} = -\frac{3}{2}. \end{cases} \\
 9. \begin{cases} \frac{12}{x} + \frac{1}{3}y = 5, \\ \frac{10}{x} - \frac{1}{2}y = -\frac{1}{2}. \end{cases} & 12. \begin{cases} 9x - \frac{1}{y} = 0, \\ 6x + \frac{3}{y} = 11. \end{cases} & 15. \begin{cases} \frac{5}{x-1} - \frac{4}{y-2} = 12, \\ \frac{4}{x-1} + \frac{3}{y-2} = 22. \end{cases} \\
 16. \begin{cases} xy + 7x = 11, \\ 5xy + 4x = 24. \end{cases} & 17. \begin{cases} \frac{2}{xy} + \frac{2}{y} = 1, \\ \frac{3}{xy} - \frac{7}{y} = -\frac{11}{3}. \end{cases}
 \end{array}$$

LITERAL SIMULTANEOUS EQUATIONS

$$202. \text{ Ex. 1. Solve } \begin{cases} ax + by = c. & (1) \\ mx + ny = p. & (2) \end{cases}$$

$$(1) \times n, \quad anx + bny = cn. \quad (3)$$

$$(2) \times b, \quad bmx + bny = bp. \quad (4)$$

$$(3) - (4), \quad anx - bmx = cn - bp.$$

$$\text{Uniting,} \quad (an - bm)x = cn - bp.$$

$$\text{Dividing,} \quad x = \frac{cn - bp}{an - bm}. \quad (5)$$

$$(1) \times m, \quad amx + bmy = cm. \quad (6)$$

$$(2) \times a, \quad amx + any = ap. \quad (7)$$

$$(7) - (6), \quad any - bmy = ap - cm.$$

$$\text{Uniting,} \quad (an - bm)y = ap - cm.$$

$$y = \frac{ap - cm}{an - bm}.$$

EXERCISE 80

Solve:

$$1. \begin{cases} ax + 2by = 6, \\ 2ax + 5by = 9. \end{cases}$$

$$2. \begin{cases} ax + by = a, \\ bx - ay = b. \end{cases}$$

$$3. \begin{cases} ax - by = 0, \\ cx - dy = 1. \end{cases}$$

$$4. \begin{cases} x + y = a, \\ x - y = b. \end{cases}$$

$$5. \begin{cases} ax + y = m, \\ x - y = n. \end{cases}$$

$$6. \begin{cases} x + my = a, \\ x - ny = b. \end{cases}$$

$$7. \begin{cases} ax + by = c, \\ dx + ey = f. \end{cases}$$

$$8. \begin{cases} ax + by = c, \\ mx = ny. \end{cases}$$

$$9. \begin{cases} a(x + y) = 16, \\ b(x - y) = 8. \end{cases}$$

$$10. \begin{cases} ax + by = 1, \\ bx + ay = 0. \end{cases}$$

$$11. \begin{cases} ax + by = 4, \\ ax - by = 2. \end{cases}$$

$$12. \begin{cases} \frac{x}{a} + \frac{y}{b} = 1, \\ \frac{x}{c} + \frac{y}{d} = 2. \end{cases}$$

$$13. \begin{cases} \frac{x+1}{y} = a, \\ \frac{y+1}{x} = b. \end{cases}$$

$$14. \begin{cases} \frac{x}{y} = \frac{a}{b}, \\ \frac{x+1}{y+1} = \frac{a+1}{b+1}. \end{cases}$$

$$15. \begin{cases} (a+b)x - (a-b)y = 4ab, \\ (a+b)x + (a-b)y = 2(a^2 + b^2). \end{cases}$$

$$16. \begin{cases} (a+c)x - (a-c)y = 2ab, \\ (a+b)y - (a-b)x = 2ac. \end{cases}$$

$$17. \begin{cases} \frac{a}{x} + \frac{1}{y} = b, \\ \frac{1}{x} - \frac{b}{y} = a. \end{cases}$$

$$19. \begin{cases} \frac{a}{x} + \frac{b}{y} = c, \\ \frac{m}{x} + \frac{n}{y} = p. \end{cases}$$

$$18. \begin{cases} \frac{a}{x} + \frac{b}{y} = 1, \\ \frac{c}{x} + \frac{d}{y} = 2. \end{cases}$$

$$20. \begin{cases} ax - by = a, \\ ab(x-1) = y. \end{cases}$$

21. Solve the equations

$$ax + by = c,$$

$$px + qy = r,$$

and from the results find by numerical substitution the roots of Exs. 1-4, page 188.

22. Find a and s in terms of n , d , and l if

$$\begin{cases} s = \frac{n}{2}(a + l), \\ l = a + (n - 1)d. \end{cases}$$

23. From the same simultaneous equations find d in terms of a , n , and l .

24. From the same equations find s in terms of a , d , and l .

25. From the same system of equations find n in terms of a , l , and s .

SIMULTANEOUS EQUATIONS INVOLVING MORE THAN TWO UNKNOWN QUANTITIES

203. To solve equations containing three unknown quantities three simultaneous independent equations must be given.

By eliminating one unknown quantity from any pair of equations, and the same unknown quantity from another pair, the problem is reduced to the solution of two simultaneous equations containing two unknown quantities.

Similarly four equations, containing four unknown quantities, are reduced to three equations containing three unknown quantities, etc.

Ex. 1. Solve the following system of equations:

$$2x - 3y + 4z = 8. \tag{1}$$

$$3x + 4y - 5z = -4. \tag{2}$$

$$4x - 6y + 3z = 1. \tag{3}$$

Eliminate y .

$$\text{Multiplying (1) by 4,} \quad 8x - 12y + 16z = 32$$

$$\text{Multiplying (2) by 3,} \quad 9x + 12y - 15z = -12$$

$$\text{Adding,} \quad \begin{array}{rcl} 17x & + & z = 20 \end{array} \quad (4)$$

$$\text{Multiplying (2) by 3,} \quad 9x + 12y - 15z = -12$$

$$\text{Multiplying (3) by 2,} \quad 8x - 12y + 6z = 2$$

$$\text{Adding,} \quad \begin{array}{rcl} 17x & - & 9z = -10 \end{array} \quad (5)$$

Eliminating x from (4) and (5).

$$(4) - (5), \quad 10z = 30.$$

$$\text{Therefore} \quad z = 3. \quad (6)$$

$$\text{Substitute this value in (4),} \quad 17x + 3 = 20.$$

$$\text{Therefore} \quad x = 1. \quad (7)$$

Substituting the values of x and z in (1),

$$2 - 3y + 12 = 8.$$

$$3y = 6.$$

$$\text{Hence} \quad y = 2. \quad (8)$$

$$\text{Check.} \quad \begin{array}{l} 2 \cdot 1 - 3 \cdot 2 + 4 \cdot 3 = 8; \quad 3 \cdot 1 + 4 \cdot 2 - 5 \cdot 3 = -4; \\ 4 \cdot 1 - 6 \cdot 2 + 3 \cdot 3 = 1. \end{array}$$

204. After some practice the student will be able to eliminate the unknown quantities of simple examples mentally. Eliminating by subtraction should then be avoided, and by the use of negative multipliers every elimination should be based upon addition.

Ex. 2. Solve	(1) $2x + 3y + 2z = 21$	3	4	Multipliers for the elimination of z .
	(2) $5x - 2y + 3z = 16$	-2	4	
	(3) $7x - 5y + 4z = 15$	-3	-3	
	(4) $-4x + 13y = 31$	1	Multipliers for the elimi- nation of x .	
	(5) $-x + 7y = 19$	-4		
$-15y = -45.$				

$$\text{Therefore} \quad y = 3.$$

$$\text{From (5),} \quad x = 2.$$

$$\text{From (1),} \quad z = 4.$$

EXERCISE 81

Solve:

$$1. \begin{cases} x + 5y + 6z = 29, \\ 10x + y + 2z = 18, \\ 5x + 9y + 3z = 32. \end{cases}$$

$$2. \begin{cases} 2x + 5y - 3z = 13, \\ 6x - 3y + 4z = 16, \\ 5x + 3y - 6z = 15. \end{cases}$$

$$3. \begin{cases} 2x + 3y - z = 21, \\ 9x + 5y - 2z = 71, \\ 6x - 7y + 5z = 55. \end{cases}$$

$$4. \begin{cases} x + 2y + 3z = 32, \\ 2x + y + 3z = 31, \\ 4x - 2y + z = 12. \end{cases}$$

$$5. \begin{cases} 2x - 3y + 4z = 8, \\ 3x + 4y - 5z = -4, \\ 4x - 6y + 3z = 1. \end{cases}$$

$$6. \begin{cases} 7x - 6y + 2z = 4, \\ 14x - 8y - z = 0, \\ 2x - 9y - 3z = -35. \end{cases}$$

$$7. \begin{cases} 6x + 5y - 4z = 98, \\ 9x - 15y + 6z = -6, \\ 8x - 3y - 9z = 12. \end{cases}$$

$$8. \begin{cases} 9x - 7y - 6z = 18, \\ 12x - 14y + 9z = 27, \\ 18x - 35y + 15z = 0. \end{cases}$$

$$9. \begin{cases} 3x - y + z = 7, \\ x + 2y - 4z = -8, \\ 2x - 2y + z = 2. \end{cases}$$

$$10. \begin{cases} x + y + z = 12, \\ 4x + 3y + 5z = 49, \\ 5x - 2y + z = 1. \end{cases}$$

$$11. \begin{cases} x + y + z = 9, \\ x - y + z = -1, \\ x + y - z = -5. \end{cases}$$

$$12. \begin{cases} 12x - 5y + 18z = 30, \\ 2x - 5y + 8z = 15, \\ 4x + y + 4z = 3. \end{cases}$$

$$13. \begin{cases} 5x + y - 4z = 0, \\ 3x + 2y + 3z = 110, \\ 2x - 3y + z = 0. \end{cases}$$

$$14. \begin{cases} 2x + 4y + 8z = 13, \\ x + y + z = 3, \\ 3x + 9y + 27z = 34. \end{cases}$$

$$15. \begin{cases} x + 2y + 3z = 32, \\ 2x + 3y + z = 42, \\ 3x + y + 2z = 40. \end{cases}$$

$$16. \begin{cases} 3x + 3y + z = 17, \\ 3x + y + 3z = 15, \\ x + 3y + 3z = 13. \end{cases}$$

$$17. \begin{cases} 2x - 3y - 4z = 6, \\ 3x - 4y + 5z = 56, \\ 5x + 7y - 9z = 106. \end{cases}$$

$$18. \begin{cases} x + 2y + 3z = 10, \\ 3x - 5y - 9z = -10, \\ 2x - 3y = 0. \end{cases}$$

$$19. \begin{cases} 5y - 7x + 4z = -44, \\ z - 5y = -33, \\ 5y - x = 22. \end{cases}$$

$$24. \begin{cases} 3x - 4y + 5z = 18, \\ x = 4z - 17, \\ y = 5z - 21. \end{cases}$$

$$20. \begin{cases} x + y = 23, \\ y + z = 25, \\ x + z = 24. \end{cases}$$

$$25. \begin{cases} \frac{x}{3} + \frac{y}{5} + \frac{z}{4} = 24, \\ \frac{x}{4} + \frac{y}{3} - \frac{z}{5} = 8, \\ \frac{x}{6} - \frac{y}{15} + \frac{z}{10} = 6. \end{cases}$$

$$21. \begin{cases} 3x + 4y = 25, \\ 5y + 6z = 50, \\ 7z + 8x = 59. \end{cases}$$

$$26. \begin{cases} 1.4x - 2.4y - 3z = -.4, \\ .8x + 3.2y + 4z = 11.2, \\ x - 2y + 1.8z = 7.6. \end{cases}$$

$$22. \begin{cases} x + y + z = 39, \\ 13x = 10y, \\ 16y = 13z. \end{cases}$$

$$23. \begin{cases} 2x + 3y - 4z = -3, \\ y = 5x - 7, \\ z = 7x - 10. \end{cases}$$

$$27. \begin{cases} .1x + .2y + .3z = 3.2, \\ .2x + .1y + .3z = 3.1, \\ .3x + .2y + .1z = 8. \end{cases}$$

$$28. \begin{cases} 5(2x - 3y) + 3(3x - 5z) = 31, \\ 2(2x - 3y) + 5(3y - 7z) = -45, \\ 4(3x - 5y) - 3(5y - 7z) = 55. \end{cases}$$

$$29. \begin{cases} \frac{x+y}{y-z} = 10, \\ \frac{x+z}{x-y} = 9, \\ \frac{y+z}{x+5} = 1. \end{cases}$$

$$30. \begin{cases} x + 3 : y + z = 2 : 1, \\ y + 3 : x + z = 1 : 1, \\ x + 3 : x + y = 1 : 2. \end{cases}$$

$$31. \begin{cases} \frac{3y-2x}{4} + \frac{4z-3x}{3} + \frac{2y-7z}{6} = 0, \\ \frac{5x-3y}{15} - \frac{4y-3x}{5} + \frac{7y+2z}{9} = 2\frac{2}{15}, \\ \frac{7z}{8} - \frac{6x-5y}{12} + \frac{3z-8x}{3} = -5\frac{23}{4}. \end{cases}$$

$$32. \begin{cases} \frac{1}{x} + \frac{2}{y} + \frac{3}{z} = 3, \\ \frac{5}{x} - \frac{4}{y} = 3, \\ \frac{6}{z} + \frac{7}{y} = 5\frac{1}{2}. \end{cases}$$

$$33. \begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 12, \\ \frac{3}{x} - \frac{4}{y} + \frac{5}{z} = 18, \\ \frac{5}{x} - \frac{3}{y} + \frac{2}{z} = 13. \end{cases}$$

$$34. \begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{5}{6}, \\ \frac{1}{x} - \frac{1}{z} = \frac{1}{4}, \\ \frac{1}{y} - \frac{1}{z} = \frac{1}{12}. \end{cases}$$

$$35. \begin{cases} 5x - y + 3z = a, \\ 5y - z + 3x = b, \\ 5z - x + 3y = c. \end{cases}$$

$$36. \begin{cases} 7x + 11y + z = a, \\ 7y + 11z + x = a, \\ 7z + 11x + y = c. \end{cases}$$

$$37. \begin{cases} x + y + z = 56, \\ x : y : z = 1 : 2 : 5. \end{cases}$$

$$38. \begin{cases} x + y = 2a, \\ y + z = 2c, \\ z + x = 2b. \end{cases}$$

$$39. \begin{cases} ax + by = 2ab, \\ by + cz = ab + c^2, \\ x + y + z = a + b + c. \end{cases}$$

$$40. \begin{cases} \frac{1}{y} + \frac{1}{z} = 2a, \\ \frac{1}{x} + \frac{1}{z} = 2b, \\ \frac{1}{x} + \frac{1}{y} = 2c. \end{cases}$$

$$41. \begin{cases} ax + by + cz = a + b, \\ x + dy = d, \\ x + z = 1. \end{cases}$$

$$42. \begin{cases} 3x + 6y + 2z + v = 2, \\ x - y - 3z - 4v = 3, \\ x + 2y - 2z - 2v = 0, \\ 2x + y - z - 3v = 5. \end{cases}$$

$$43. \begin{cases} x + y + z + v = 4, \\ 2x + y - 2z = 0, \\ 3x + 4y + 3v = 1, \\ 4v - 5x + 3z + y = 1. \end{cases}$$

$$44. \begin{cases} \frac{x+y}{xy} = 5, \\ \frac{x+z}{xz} = 6, \\ \frac{y+z}{yz} = 7. \end{cases}$$

$$45. \begin{cases} a^x \cdot a^y \cdot a^z = a^9, \\ a^x \cdot a^{2y} = a^8, \\ x + z = 6. \end{cases}$$

PROBLEMS LEADING TO SIMULTANEOUS EQUATIONS

205. Problems involving several unknown quantities must contain, either directly or implied, as many verbal statements as there are unknown quantities. If the problem is to be solved by one equation containing only one unknown quantity, whose symbol is x , all statements except one are used to express the unknown quantities in terms of x , while the remaining statement produces the equation. (§ 101.)

In complex examples, however, it is often very difficult to express some of the unknown quantities in terms of one of them. In such a case it is advisable to represent every unknown quantity by a different letter, and to express every verbal statement as an equation.

Ex. 1. The sum of the three digits of a number is 8. The digit in the tens' place is $\frac{1}{3}$ of the sum of the other two digits, and if 396 be added to the number, the first and the last digit will be interchanged. Find the number.

Obviously it is very difficult to express two of the required digits in terms of the other; hence we employ 3 letters for the three unknown quantities.

Let x = the digit in the hundreds' place,
 y = the digit in the tens' place,
 and z = the digit in the units' place.

Then $100x + 10y + z$ = the number.

The three statements of the problem can now be readily expressed in symbols:

$$x + y + z = 8. \quad (1)$$

$$y = \frac{1}{3}(x + z). \quad (2)$$

$$100x + 10y + z + 396 = 100z + 10y + x. \quad (3)$$

The solution of these equations gives $x = 1$, $y = 2$, $z = 5$.
 Hence the required number is 125.

Check. $1 + 2 + 5 = 8$; $2 = \frac{1}{3}(1 + 5)$; $125 + 396 = 521$.

206. Between the two methods of expressing all unknown quantities in terms of one letter, x , and employing as many

letters as there are unknown quantities, there is often the intermediate way of representing some, but not all, unknown quantities in different letters. *E.g.* denote two quantities directly by letters, as x and y , and express the remaining unknown quantities in terms of x and y by some of the given verbal statements. The remaining two statements give the two necessary equations.

E.g. in the example of the preceding paragraph, we could make x = the first digit, and y = the last digit. Then, according to the second statement, the second digit = $\frac{1}{3}(x + y)$, and the first and third verbal statements, expressed in symbols, give

$$x + \frac{1}{3}(x + y) + y = 8.$$

$$100x + \frac{10}{3}(x + y) + y + 396 = 100y + \frac{10}{3}(x + y) + x.$$

These equations lead of course to the same number as before.

Although this method is often somewhat shorter than the method previously described, it is advantageous only if the example is simple. In most complex examples, the equations can be written with the least difficulty if each unknown quantity is represented by a different letter.

Ex. 2. If both numerator and denominator of a fraction be increased by one, the fraction is reduced to $\frac{2}{3}$; and if both numerator and denominator of the reciprocal of the fraction be diminished by one, the fraction is reduced to 2. Find the fraction.

Let x = the numerator,
and y = the denominator.

then $\frac{x}{y}$ = the fraction. By expressing the two statements in symbols, we obtain,

$$\frac{x+1}{y+1} = \frac{2}{3}, \quad (1)$$

and $\frac{y-1}{x-1} = 2. \quad (2)$

These equations give $x = 3$ and $y = 5$. Hence the fraction is $\frac{3}{5}$.

Check. $\frac{3+1}{5+1} = \frac{4}{6} = \frac{2}{3}; \quad \frac{5-1}{3-1} = \frac{4}{2} = 2.$

Ex. 3. A, B, and C travel from the same place in the same direction. B starts two hours after A and travels one mile per hour faster than A. C, who travels 2 miles an hour faster than B, starts 2 hours after B and overtakes A at the same instant as B. How many miles has A then traveled?

	TIME (Hours)	RATE (Miles per hour)	DISTANCE (Miles)
A	x	y	xy
B	$x - 2$	$y + 1$	$xy + x - 2y - 2$
C	$x - 4$	$y + 3$	$xy + 3x - 4y - 12$

Since the three men traveled the same distance,

$$xy = xy + x - 2y - 2. \quad (1)$$

$$xy = xy + 3x - 4y - 12. \quad (2)$$

Or $x - 2y = 2. \quad (3)$

$$3x - 4y = 12. \quad (4)$$

$$(4) - 2 \times (3) \quad x = 8.$$

$$\text{From (3)} \quad y = 3.$$

Hence $xy = 24$ miles, the distance traveled by A.

Check. $8 \times 3 = 24$, $6 \times 4 = 24$, $4 \times 6 = 24$.

EXERCISE 82

1. Three times a certain number increased by five times another number equals 31, and the second one increased by one is equal to three times the first one. Find the numbers.

2. Find two numbers whose sum and whose quotient equal 5.

3. Half the sum of two numbers is 11, and the fifth part of their difference is 2. Find the numbers.

4. If 1 be added to the numerator of a fraction, its value is $\frac{1}{2}$. If 1 be added to its denominator, the fraction is reduced to $\frac{1}{3}$. Find the fraction. (See Ex. 2, § 206.)

5. If 3 be added to both terms of a fraction, its value is $\frac{1}{2}$. If 1 be subtracted from both terms, its value is $\frac{1}{6}$. Find the fraction.

6. If the numerator of a fraction is trebled, and its denominator diminished by 2, it is reduced to $\frac{5}{2}$. If the denominator is doubled, and the numerator increased by 3, the fraction is reduced to $\frac{1}{2}$. Find the fraction.

7. A fraction is reduced to $\frac{1}{2}$, if both its terms are increased by 1, and 8 times the numerator diminished by 3 times the denominator equals 3. What is the fraction?

8. Find two fractions, with numerators 2 and 3 respectively, whose sum is $\frac{17}{12}$, and such that when their denominators are interchanged, their sum is $1\frac{1}{2}$.

9. The sum of the digits of a number of two figures is 10, and if 3 times the units' digit is added to the number, the digits will be interchanged. What is the number? (See Ex. 1, § 205.)

10. The sum of a number of two digits and of the number formed by reversing the digits is 121, and five times the tens' digit exceeds the units' digit by one. Find the number.

11. The sum of the three digits of a number is 11, and the sum of the first two digits exceeds the last digit by 1. If 27 is added to the number, the last two digits are interchanged. Find the number.

12. A number consists of three digits, the sum of the last two digits is two less than five times the first digit; three times the second digit exceeds the first digit by one; and if the first digit be subtracted from the number, the remainder is 215. Find the number.

13. If a certain number be divided by the sum of its three digits, the quotient is 43. The sum of the first two digits is equal to the third digit, and if 99 be added to the number, the first and last digits will be interchanged. Find the number.

14. If a certain number be divided by the sum of its two digits, the quotient is 4 and the remainder 3. Three times the first digit exceeds the second digit by 3. Find the number.

15. Twice A's age exceeds the sum of B's and C's ages by 30, and B's age is $\frac{1}{2}$ the sum of A's and C's ages. Ten years ago the sum of their ages was 60. Find their present ages.

16. Five years ago A was as old as B will be in 10 years; and 10 years ago B was as old as C will be in 5 years. If the sum of their ages is 60, how old now is each?

17. A sum of \$10,000 is partly invested at 5%, partly at 4%, and partly at 3%, bringing a total yearly interest of \$390. The 5% investment brings \$10 more interest than the 4% and 3% investments together. How much money is invested at 3%, 4%, and 5% respectively?

18. A sum of money at simple interest amounted in 2 years to \$330, in 5 years to \$375. What was the sum and the rate of interest?

19. A sum of money at simple interest amounted in 6 years to \$16,000, and in 8 years to \$17,000. What was the sum of money and the interest?

20. A sum of money at 6% simple interest amounted in a certain time to \$944. In half the time at 4% it would amount to \$348. Find the sum and the time.

21. A sum of money at simple interest amounted in a years to m dollars, and in b years to n dollars. What was the sum? Find the sum if $a = 3$, $b = 5$, $m = \$500$, and $n = \$600$.

22. The sums of \$ 1200 and \$ 1400 are invested at different rates and their annual interest is \$ 111. If the rates of interest were exchanged, the annual interest would be \$ 110. Find the rates of interest. ✕

23. Three sums of money are respectively invested at 4%, 4%, and 5%, and their annual interest is \$1280. If the first and third were invested at 4%, and the second at 5%, the interest of the second would be \$ 350 less than the interest of the other two. If the rates of interest were respectively 5%, 4%, and $3\frac{1}{2}\%$, the first sum would bring \$ 380 less interest than the other two. What are the sums? ✕

24. If a rectangle has the same area as another 3 feet longer and 2 feet narrower, and the same area as a third rectangle which is 8 feet longer and 4 feet narrower, what are its dimensions? ✕

25. If a rectangle were 100 feet longer and 25 feet narrower, its area would contain 2500 square feet more. If it were 100 feet shorter and 50 feet wider, its area would contain 5000 square feet less. What are the dimensions of the rectangle?

26. If a rectangle were 1 foot longer and 1 foot narrower, its area would be m square feet less. If it were 1 foot longer and one foot wider, its area would be n feet more. What are the dimensions of the rectangle?

27. A, B, and C working together can do a piece of work in 1 day; A and C together in $1\frac{1}{2}$ days, and B and C together in 2 days. In how many days can each do the same work alone? ✕

28. A and B together do a piece of work in 18 days; A and C in 16 days, and B and C in 20 days. In how many days can each alone do the same work? ✕

29. A and B together can do a piece of work in a days; B and C in b days, and C and A in c days. In how many days can each alone do the same work? Find the answer if $a=2$, $b=3$, $c=4$.

30. A farmer sold a number of horses, cows, and sheep, for \$500, receiving \$100 for each horse, \$50 for each cow, and \$15 for each sheep. The number of sheep was twice the number of horses and cows together. How many did he sell of each if the total number of animals was fifteen?

31. The sum of the 3 angles of any triangle is 180° . If one angle of a triangle exceeds half the sum of the other two angles by 15° and half their difference by 65° , what are the angles?

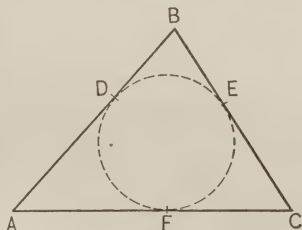
32. The difference of two angles of a triangle is equal to the third angle, and their sum is $\frac{1}{3}$ of the third angle. What are the angles?

33. A and B received together \$107 wages for working 25 and 16 days respectively. If A had worked 24 days and B had worked 20 days, they would have received \$112. What were the daily wages of each?

34. The perimeter (*i.e.* the sum of the sides) of a triangle is 39 inches. The greatest side is 7 inches less than the sum of the other two, and one of these two is twice as large as the difference of the remaining two. Find the length of each side.

35. On the three sides of a triangle ABC , respectively, three points D , E , and F , are taken so that $AD = AF$, $BD = BE$, and $CE = CF$. If $AB = 6$ inches, $BC = 4$ inches, and $AC = 8$ inches, what is the length of AD , BE , and CF ?

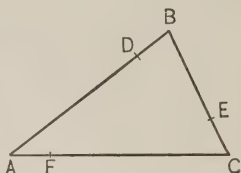
NOTE. If a circle is inscribed in the triangle ABC touching the sides in D , E , and F (see diagram), then $AD = AF$, $BD = BE$, and $CE = CF$.



36. A circle is inscribed in triangle ABC touching the three sides in D , E , and F . Find the parts of the sides if $AB = 5$, $BC = 7$, and $CA = 8$.

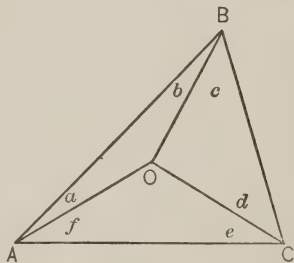
37. A circle is inscribed in a triangle ABC (see diagram of Ex. 35) whose perimeter is 14 feet, AD exceeds FC by 2 inches, and BD equals 1 inch. Find the sides of the triangle.

38. On the sides of a triangle ABC the points D , E , and F are so taken that AD is 4 times as large as AF , BE is 3 times as large as BD , and CF is 5 times as large as CE . If $AB = 5$ inches, $BC = 4$ inches, and $CA = 6$ inches, find the parts of the three sides.



39. In the annexed diagram angle $a =$ angle b , angle $c =$ angle d , and angle $e =$ angle f . If angle $ABC = 60^\circ$, angle $BAC = 40^\circ$, and angle $BCA = 80^\circ$, find angles a , c , and e .

NOTE. O is the center of the circumscribed circle.



40. The sum of the radii of two circles is 15 inches, and the difference of their circumferences is 44. If π is taken equal to $3\frac{1}{7}$, what are the radii? (The circumference of a circle C , whose radius is R , is determined by the formula $C = 2\pi R$.)

41. The sum of the radii of two circles is r inches, and the difference of their circumferences is d inches. Find the radii.

42. Two persons start to walk in the same direction from two stations 12 miles apart, and one overtakes the other after 6 hours. If they had walked toward each other, they would have met in 2 hours. What are their rates of travel?

43. Two persons start to walk in the same direction from two stations d miles apart, and one overtakes the other after a hours. If they had walked toward each other, they would have met in b hours. What are their rates of travel?

44. A takes 2 hours longer than B to travel 12 miles, but if A would double his pace, he would walk it in one hour less than B. Find their rates of walking.

45. A takes 2 hours longer than B to travel d miles, but if A should double his pace, he would walk it in 1 hour less. Find the time B needs to walk the distance.

INTERPRETATION OF NEGATIVE RESULTS AND THE

FORMS OF $\frac{0}{0}$, $\frac{a}{0}$, $\frac{0}{\infty}$.

207. The results of problems and other examples appear sometimes in forms which require a special interpretation, as

$$\frac{a}{0}, \frac{0}{0}, \frac{0}{\infty}, \text{ etc.}$$

208. Interpretation of $\frac{0}{0}$. According to the definition of division, $\frac{0}{0} = x$, if $0 = 0x$. But this equation is satisfied by any finite value of x , hence $\frac{0}{0}$ may be any finite number, or $\frac{0}{0}$ is indeterminate.

209. Interpretation of $\frac{a}{0}$. The fraction $\frac{a}{x}$ increases if x decreases; *e.g.* $\frac{a}{\frac{1}{100}} = 100a$, $\frac{a}{\frac{1}{10000}} = 10,000a$. By making x sufficiently small, $\frac{a}{x}$ can be made larger than any assigned number, however great. If x approaches the value zero, $\frac{a}{x}$ becomes infinitely large. It is customary to represent this result by the equation $\frac{a}{0} = \infty$.

NOTE. The symbol ∞ is called infinity. In some examples, the result $\frac{0}{0}$ is only indeterminate in form; *e.g.* If $x = 3$, $\frac{x^2 - 4x + 3}{x^2 - 5x + 6}$ would be $\frac{0}{0}$, if we substitute directly. By reducing the fraction to its lowest terms, $\frac{(x-3)(x-1)}{(x-3)(x-2)} = \frac{x-1}{x-2}$, and then substituting, we obtain $\frac{2}{1}$ or 2.

210. Interpretation of $\frac{a}{\infty}$. The fraction $\frac{a}{x}$ decreases if x increases, and becomes infinitely small, if x is infinitely large. This result is usually written :

$$\frac{a}{\infty} = 0.$$

211. The discussion of a problem determines the nature of the solutions, if the given quantities assume all possible values. The following example illustrates the discussion of a problem :

Two couriers, A and B, travel by the same road in the same direction, and at 12 o'clock B is d miles in advance of A. If A travels a miles per hour, and B travels b miles per hour, after how many hours will A overtake B ?

Suppose they meet after x hours, then A has traveled ax , and B bx , miles. But since A has traveled d miles more than B,

$$ax - bx = d,$$

therefore,

$$x = \frac{d}{a - b}.$$

Discussion. 1. If a is greater than b , the value of x is positive, and A will overtake B after 12 o'clock.

2. If a is smaller than b , the value of x is negative, *e.g.* If $a = 2$, $b = 3$, $d = 4$, then $x = -4$, *i.e.* the men do not meet after 12 o'clock, but they were together 4 hours before 12 o'clock. This is obvious from the data of the problem, for if A walks more slowly than B, he cannot overtake B.

Hence there is no answer to the problem stated above. To make a solution possible the problem should read, How many hours before 12 o'clock did they meet ?

3. If $a = b$, then $x = \frac{d}{0} = \infty$; *i.e.* A and B will never meet, and evidently two men traveling at the same rate, and d miles apart, will never meet. Hence the problem has no solution.

4. If $a = b$, $d = 0$, then $x = \frac{0}{0} =$ any finite number, as 2, 3, 4, etc., *i.e.* A and B are always together. This also is obvious from the nature of the problem.

212. *Negative solutions frequently indicate a fault in the enunciation of the problem.*

213. *The result $\frac{a}{0}$ or ∞ indicates that the problem has no solution. If in an equation all terms containing the unknown quantity cancel, while the remaining terms do not cancel, the root is infinity.*

214. *The solution $x = \frac{0}{0}$ indicates that the problem is indeterminate, or that x may equal any finite number. If all terms of an equation, without exception, cancel, the answer is indeterminate.*

EXERCISE 83

Interpret the answers of the following problems; and if negative solutions occur, indicate what changes in the statement of the problem would make a solution possible.

1. A is 25 years old, and B is 15 years old. How many years hence will A be twice as old as B?

2. Four times a certain number increased by 12 equals four times the excess of the number over 2. Find the number.

3. One half of a certain number exceeds the sum of its third and sixth part by 12. Find the number.

4. Find 3 consecutive numbers such that the square of the second exceeds the product of the first and third by 1.

5. One half of a certain number is equal to the sum of its fourth, sixth, and twelfth part. Find the number.

6. a times a certain number increased by b equals c times the number increased by d . Find the general answer and interpret the answer, if

(a) $a = c$, b and d are unequal.

(b) $b = d$, a and c are unequal.

(c) $a = c$, and $b = d$.

CHAPTER XII

INVOLUTION

215. Involution is the operation of raising a quantity to a positive integral power.

To find $(3a^5b)^n$ is a problem of involution. Since a power is a special kind of product, involution may be effected by repeated multiplication.

216. Law of Signs. According to § 50.

$$+a \cdot +a \cdot +a = +a^3.$$

$$-a \cdot -a \cdot = +a^2.$$

$$-a \cdot -a \cdot -a = -a^3 \text{ etc.}$$

Obviously it follows that

1. *All powers of a positive quantity are positive.*
2. *All even powers of a negative quantity are positive.*
3. *All odd powers of a negative quantity are negative.*

$(-a)^6$ is positive, $(-ab^2)^9$ is negative.

INVOLUTION OF MONOMIALS

217. According to § 52.

$$1. (a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^{2+2+2} = a^6.$$

$$2. (b^5)^4 = b^5 \cdot b^5 \cdot b^5 \cdot b^5 = b^{5+5+5+5} = b^{20}.$$

$$3. (a^n)^m = a^n \cdot a^n \dots \text{to } m \text{ factors} \\ = a^{n+n+n \text{ to } m \text{ terms}} \\ = a^{mn}.$$

$$4. (-3a^2b^3)^4 = (-3a^2b^3) \cdot (-3a^2b^3) \cdot (-3a^2b^3) \cdot (-3a^2b^3) \\ = 81a^8b^{12}.$$

$$5. \left(-\frac{2m^2}{3n^5}\right)^3 = -\frac{(2m^2)^3}{(3n^5)^3} = -\frac{8m^6}{27n^{15}}.$$

To find the exponent of the power of a power, multiply the given exponents.

To raise a product to a given power, raise each of its factors to the required power.

To raise a fraction to a power, raise its terms to the required power.

EXERCISE 84

Perform the operations indicated :

- | | | |
|-------------------------------------|--|---|
| 1. $(a^2)^4$. | 20. $(-2 a^m b^n)^6$. | 33. $\left(-\frac{7 m^2 n^2}{8 p^2 q^2}\right)^3$. |
| 2. $(-a^3)^{11}$. | 21. $(-3 a^2 b^3 c^m)^5$. | 34. $\left(\frac{2}{x^2 y^2 z^2}\right)^5$. |
| 3. $(xy^2)^4$. | 22. $(a^2)^m$. | 35. $\left(\frac{-1}{3 x^2 y^2 z^4}\right)^4$. |
| 4. $(-xy^2)^6$. | 23. $(-a^2)^{2m}$. | 36. $\left(\frac{-1}{a^m}\right)^4$. |
| 5. $(a^3 b^3)^4$. | 24. $(a^m)^n$. | 37. $\left(\frac{-1}{2 a^m b^n}\right)^3$. |
| 6. $(-a^4 b^2)^2$. | 25. $\left(\frac{2a}{3}\right)^2$. | 38. $\left(-\frac{4}{a^m b^n}\right)^3$. |
| 7. $(3mn^2)^2$. | 26. $\left(-\frac{2a}{3b}\right)^3$. | 39. $\left(\frac{6 a^m b^n c}{5 p^2 q^3}\right)^2$. |
| 8. $(-5p^2 q^3)^3$. | 27. $\left(-\frac{3a^2}{b}\right)^4$. | 40. $\left(-\frac{2a^m}{c^n}\right)^4$. |
| 9. $(-2pq^4)^6$. | 28. $\left(\frac{8 a^2 b^2 c^2}{7 x^3}\right)^2$. | 41. $\left(-\frac{2a^m}{b^n}\right)^{2n}$. |
| 10. $(-4a^4 p^3 c)^3$. | 29. $\left(\frac{a^5 b^6}{c^7 d^8}\right)^{10}$. | 42. $\left(\frac{2a^m}{b^n}\right)^m$. |
| 11. $(2x^2 y^2 z^2)^7$. | 30. $\left(\frac{2 a^2 b^2 c^3}{x^4}\right)^5$. | 43. $\left[\left(\frac{p^2}{q^3}\right)^2\right]^n$. |
| 12. $(-3ab^2 c^3 d^4)^4$. | 31. $\left(\frac{-2ab}{3 c^3 d}\right)^4$. | |
| 13. $(a^4 b^4 c^6)^5$. | 32. $\left(\frac{4 x^2 y^2 z^2}{5 pq}\right)^3$. | |
| 14. $(7abcd^{11})^2$. | | |
| 15. $(\frac{1}{2} a^2 b^2 c^2)^3$. | | |
| 16. $(-\frac{2}{3} ab^3 c^4)^2$. | | |
| 17. $(a^m)^3$. | | |
| 18. $(-x^{2n})^2$. | | |
| 19. $(-a^{3n})^3$. | | |

INVOLUTION OF BINOMIALS

218. The square of a binomial was discussed in § 65.

$$(a + b)^2 = a^2 + 2ab + b^2.$$

219. The cube of a binomial we obtain by multiplying $(a + b)^2$ by $a + b$.

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

and
$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

Ex. 1. Find the cube of $2x + 3y$.

$$\begin{aligned}(2x + 3y)^3 &= (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3 \\ &= 8x^3 + 36x^2y + 54xy^2 + 27y^3.\end{aligned}$$

Ex. 2. Find the cube of $3x^2 - y^n$.

$$\begin{aligned}(3x^2 - y^n)^3 &= (3x^2)^3 - 3(3x^2)^2(y^n) + 3(3x^2)(y^n)^2 - (y^n)^3 \\ &= 27x^6 - 27x^4y^n + 9x^2y^{2n} - y^{3n}.\end{aligned}$$

EXERCISE 85

Perform the operations indicated:

- | | | |
|---------------------|-------------------------|----------------------------|
| 1. $(m + n)^3$. | 9. $(3a + 1)^3$. | 17. $(4x^2 - 5y^2)^3$. |
| 2. $(x - y)^3$. | 10. $(5a^2 + 1)^3$. | 18. $(3a^3 - 2bc)^3$. |
| 3. $(x - 7)^3$. | 11. $(7a^3 - 1)^3$. | 19. $(5x^2y^2 - 3z^2)^3$. |
| 4. $(x + y)^3$. | 12. $(5x + 2y)^3$. | 20. $(4xyz - 1)^3$. |
| 5. $(1 + 2x)^2$. | 13. $(3x - 5y)^2$. | 21. $(2x - y^m)^3$. |
| 6. $(1 + 2x)^3$. | 14. $(3x^2y^2 - 1)^3$. | 22. $(x^m - y^n)^3$. |
| 7. $(1 - 3x^2)^2$. | 15. $(ax + by)^3$. | 23. $(2a^n - 3)^3$. |
| 8. $(1 - 3x^2)^3$. | 16. $(2ax - 3by)^2$. | 24. $(2a^m - 3b^n)^3$. |

220. The higher powers of binomials, frequently called expansions, are obtained by multiplication, as follows:

$$(a + b)^2 = a^2 + 2ab + b^2.$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5, \text{ etc.}$$

An examination of these results shows that:

1. *The number of terms is one greater than the exponent of the binomial.*

2. *The exponent of a in the first term is the same as the exponent of the binomial, and decreases in each succeeding term by one.*

3. *The exponent of b is 1 in the second term of the result, and increases by 1 in each succeeding term.*

4. *The coefficient of the first term is 1.*

5. *The coefficient of the second term equals the exponent of the binomial.*

6. *The coefficient of any term of the power multiplied by the exponent of a , and the result divided by 1 plus the exponent of b , is the coefficient of the next term.*

Ex. 1. Expand $(x + y)^5$.

$$\begin{aligned}(x + y)^5 &= x^5 + 5x^4y + \frac{5 \cdot 4}{1 \cdot 2}x^3y^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3}x^2y^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4}xy^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}y^5 \\ &= x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.\end{aligned}$$

Ex. 2. Expand $(x - y)^5$.

$$\begin{aligned}(x - y)^5 &= x^5 + 5x^4(-y) + 10x^3(-y)^2 + 10x^2(-y)^3 + 5x(-y)^4 \\ &\quad + (-y)^5 \\ &= x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5.\end{aligned}$$

221. The signs of the last answer are alternately plus and minus, since the even powers of $-y$ are positive, and the odd powers negative.

Ex. 3. Expand $(2x^2 - 3y^3)^4$.

$$\begin{aligned}(2x^2 - 3y^3)^4 &= (2x^2)^4 - 4(2x^2)^3(3y^3) + 6(2x^2)^2(3y^3)^2 \\ &\quad - 4(2x^2)(3y^3)^3 + (3y^3)^4 \\ &= 16x^8 - 96x^6y^3 + 216x^4y^6 - 216x^2y^9 + 81y^{12}.\end{aligned}$$

EXERCISE 86

Expand:

- | | | |
|--------------------|-------------------|------------------------|
| 1. $(a-b)^4$. | 14. $(m+n)^8$. | 27. $(mn+1)^6$. |
| 2. $(m+1)^4$. | 15. $(m-n)^8$. | 28. $(a+bc)^4$. |
| 3. $(1-n)^4$. | 16. $(a-x)^6$. | 29. $(a^2+b^2c^2)^5$. |
| 4. $(ab+c)^4$. | 17. $(m-n)^7$. | 30. $(2a-1)^3$. |
| 5. $(x^2-y^2)^4$. | 18. $(1+a)^5$. | 31. $(2a-1)^5$. |
| 6. $(1+a^2)^4$. | 19. $(a-1)^5$. | 32. $(1+2x)^4$. |
| 7. $(m-n)^5$. | 20. $(1-a)^5$. | 33. $(3a^2+5)^3$. |
| 8. $(a+x)^5$. | 21. $(1-a)^9$. | 34. $(3a^2-5)^4$. |
| 9. $(x-b)^5$. | 22. $(a-1)^8$. | 35. $(2x^2-5y^2)^4$. |
| 10. $(b+x)^4$. | 23. $(a-2)^8$. | 36. $(2a+5c^4)^3$. |
| 11. $(a+b)^6$. | 24. $(1-m^2)^3$. | 37. $(x^2+5y^2)^4$. |
| 12. $(x-n)^6$. | 25. $(1-m^2)^4$. | 38. $(3x+2y)^4$. |
| 13. $(x+y)^7$. | 26. $(m^3+1)^5$. | 39. $(2x^2-5y^2)^5$. |

INVOLUTION OF POLYNOMIALS

222. The square of polynomials was discussed in § 67.

223. The higher powers of polynomials are found either by multiplication, or by transforming the polynomials into binomials.

Ex. 1. Expand $(a+b-c)^3$.

$$\begin{aligned}
 (a+b-c)^3 &= [(a+b)-c]^3 \\
 &= (a+b)^3 - 3(a+b)^2c + 3(a+b)c^2 - c^3 \\
 &= a^3 + 3a^2b + 3ab^2 + b^3 - 3c(a^2 + 2ab + b^2) + 3ac^2 + 3bc^2 - c^3 \\
 &= a^3 + 3a^2b + 3ab^2 + b^3 - 3a^2c - 6abc - 3b^2c + 3ac^2 + 3bc^2 - c^3.
 \end{aligned}$$

Ex. 2. Expand $(x^3 - 3x^2 - 2x - 1)^3$.

$$\begin{aligned}
 & (x^3 - 3x^2 - 2x - 1)^3 \\
 = & [(x^3 - 3x^2) - (2x + 1)]^3 \\
 = & (x^3 - 3x^2)^3 - 3(x^3 - 3x^2)^2(2x + 1) + 3(x^3 - 3x^2)(2x + 1)^2 - (2x + 1)^3 \\
 = & x^9 - 9x^8 + 27x^7 - 27x^6 - 3(x^6 - 6x^5 + 9x^4)(2x + 1) + 3(x^3 - 3x^2)(4x^2 + 4x + 1) \\
 & - (8x^3 + 12x^2 + 6x + 1) \\
 = & x^9 - 9x^8 + 27x^7 - 27x^6 - 6x^7 + 33x^6 - 36x^5 - 27x^4 + 12x^5 - 24x^4 \\
 & - 33x^3 - 9x^2 - 8x^3 - 12x^2 - 6x - 1 \\
 = & x^9 - 9x^8 + 21x^7 + 6x^6 - 24x^5 - 51x^4 - 41x^3 - 21x^2 - 6x - 1.
 \end{aligned}$$

EXERCISE 87

Expand:

- | | |
|-------------------------------------|--|
| 1. $(a + b + c)^2$. | 10. $(x^2 - x - 1)^3$. |
| 2. $(a + b - c - d)^2$. | 11. $(a + b + c + d)^3$. |
| 3. $(x - y - 3c + 5d)^2$. | 12. $(a - b + c - d)^3$. |
| 4. $(m^2 - 2n^2 - 3p^2 + 4q^2)^2$. | 13. $(a - b - c - d)^3$. |
| 5. $(x^3 - 3x^2 + 4x - 1)^2$. | 14. $(x^3 + x^2 + x + 1)^3$. |
| 6. $(a + b + c)^3$. | 15. $(a^2 + a + 1)^4$. |
| 7. $(a - b + c)^3$. | 16. $\left(a + 1 + \frac{1}{a}\right)^3$. |
| 8. $(x + y - z)^3$. | |
| 9. $(x^2 + x + 1)^3$. | |

CHAPTER XIII

EVOLUTION

224. **Evolution** is the operation of finding a root of a quantity ; it is the inverse of involution.

$$\sqrt[n]{a} = x \text{ means } x^n = a.$$

$$\sqrt[3]{27} = y \text{ means } y^3 = 27, \text{ or } y = 3.$$

$$\sqrt[5]{b^{20}} = x \text{ means } x^5 = b^{20}, \text{ or } x = b^4.$$

225. It follows from the law of signs in involution that :

1. *Any even root of a positive quantity may be either positive or negative.*

2. *Every odd root of a quantity has the same sign as the quantity.*

$$\sqrt{9} = +3, \text{ or } -3 \text{ (usually written } \pm 3\text{); for } (+3)^2 \text{ and } (-3)^2 \text{ equal } 9.$$

$$\sqrt[3]{-27} = -3, \text{ for } (-3)^3 = -27.$$

$$\sqrt[4]{a^4} = \pm a, \text{ for } (+a)^4 = a^4, \text{ and } (-a)^4 = a^4.$$

$$\sqrt[5]{32} = 2, \text{ etc.}$$

226. Since even powers can never be negative, it is evidently impossible to express an even root of a negative quantity by the usual system of numbers. Such roots are called **imaginary numbers**, and all other numbers are, for distinction, called **real numbers**.

Thus $\sqrt{-1}$ is an imaginary number, which can be simplified no further.

EVOLUTION OF MONOMIALS

227. The following examples are solved by the definition of a root:

Ex. 1. $\sqrt[4]{a^{12}} = \pm a^3$, for $(\pm a^3)^4 = a^{12}$.

Ex. 2. $\sqrt[n]{a^{mn}} = a^m$, for $(a^m)^n = a^{mn}$.

Ex. 3. $\sqrt[3]{8 a^6 b^3 c^{12}} = 2 a^2 b c^4$, for $(2 a^2 b c^4)^3 = 8 a^6 b^3 c^{12}$.

Ex. 4. $\sqrt[4]{\frac{81 a^{12} b^4}{c^4}} = \pm \frac{3 a^3 b}{c}$, for $\left(\pm \frac{3 a^3 b}{c}\right)^4 = \frac{81 a^{12} b^4}{c^4}$.

Ex. 5. $\sqrt[5]{-\frac{32 a^{15}}{243 b^{10} c^{20}}} = -\frac{2 a^3}{3 b^2 c^4}$, for $\left(-\frac{2 a^3}{3 b^2 c^4}\right)^5 = -\frac{32 a^{15}}{243 b^{10} c^{20}}$.

228. To extract the root of a power, divide the exponent by the index.

A root of a product equals the product of the roots of the factors.

To extract a root of a fraction, extract the roots of the numerator and denominator.

EXERCISE 88

Simplify the following expressions:

- | | | |
|--------------------------|---|---|
| 1. $\sqrt[3]{7^3}$. | 12. $\sqrt[3]{27 a^3 b^3 c^6}$. | 21. $\sqrt[3]{\frac{-a^6}{64 c^3 x^3}}$. |
| 2. $\sqrt{25}$. | 13. $\sqrt[3]{-27}$. | 22. $\sqrt[5]{-\frac{32 x^{10} y^{10}}{a^5}}$. |
| 3. $\sqrt{a^4}$. | 14. $\sqrt[3]{-27 a^3 b^3}$. | 23. $\sqrt[3]{-3\frac{3}{8} a^3}$. |
| 4. $\sqrt[3]{a^{12}}$. | 15. $\sqrt[3]{-512 a^{30} b^{15}}$. | 24. $\sqrt[10]{\frac{a^{10} b^{10}}{c^{20}}}$. |
| 5. $\sqrt[5]{a^{25}}$. | 16. $\sqrt{144 x^{10} y^{10}}$. | 25. $\sqrt[n]{a^n b^{2n}}$,
n being even. |
| 6. $\sqrt{5^4}$. | 17. $\sqrt[4]{16 a^8 b^4}$. | 26. $\sqrt[n]{2^{4n}}$,
n being odd. |
| 7. $\sqrt[3]{7^6}$. | 18. $\sqrt[10]{2^{20} a^{10}}$. | 27. $\sqrt[5]{\frac{32 a^{5m} b^c}{c^{10m}}}$. |
| 8. $\sqrt{100}$. | 19. $\sqrt[3]{\frac{a^3 b^9}{c^6}}$. | |
| 9. $\sqrt[3]{-1000}$. | 20. $\sqrt[4]{\frac{16 a^4}{81 b^8}}$. | |
| 10. $\sqrt{4 a^2 b^4}$. | | |
| 11. $\sqrt{9 x^6 y^8}$. | | |

28. $\sqrt[5]{(-a)^5}$. 32. $\sqrt{(a+b+c)^4}$. 36. $\sqrt[2]{(a+b)^{2x}}$,
 x being even
 29. $(\sqrt[3]{-a})^6$. 33. $\sqrt{a^2x^4(x+y)^4}$. 37. $\sqrt{\sqrt[3]{a^{12}}}$.
 30. $\sqrt{(a+b)^2}$. 34. $\sqrt[3]{-8(x-y)^6}$. 38. $\sqrt[3]{\sqrt[3]{-a^{27}}}$.
 31. $\sqrt{a^2+2ab+b^2}$. 35. $\sqrt[5]{-32(m+n)^{5p}}$. 39. $\sqrt{49p^2}$.
 40. $\sqrt[3]{.027a^{3m}}$. 42. $\sqrt[3]{27a^3}-\sqrt[3]{-8a^3}-\sqrt[3]{a^3}$.
 41. $\sqrt{16a^4}+\sqrt[3]{64a^6}$. 43. $\sqrt{4a^4b^6}-\sqrt[3]{8a^6b^9}$.

EVOLUTION OF POLYNOMIALS AND ARITHMETICAL NUMBERS

229. A **trinomial** is a perfect square if one of its terms is equal to twice the product of the square roots of the two other terms. (§ 118.) In such a case the square root can be found by inspection.

Ex. 1. Find the square root of $x^6 - 6x^3y^2 + 9y^4$.

$$x^6 - 6x^3y^2 + 9y^4 = (x^3 - 3y^2)^2. \quad (\S 118.)$$

Hence $\sqrt{x^6 - 6x^3y^2 + 9y^4} = \pm (x^3 - 3y^2)$.

EXERCISE 89

Extract the square roots of the following expressions :

1. $1 - 4a + 4a^2$. 7. $4a^2 - 44ab^2 + 121b^4$.
2. $a^4 + 16b^2 - 8a^2b$. 8. $16x^4 - 8x^2y^2 + y^4$.
3. $a^4 + 1 - 2a^2$. 9. $a^4 + b^8 - 2a^2b^4$.
4. $16a^4 + a^6 + 8a^5$. 10. $16a^4 - 120a^2bc + 225b^2c^2$.
5. $1 + 49y^4 - 14y^2$. 11. $49m^6 - 140m^3n^2 + 100n^4$.
6. $x^2y^2 - 6xyz + 9z^2$. 12. $81x^4y^2 - 126x^2yz^3 + 49z^6$.
13. $a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$.
14. $a^2 + b^2 + c^2 + 2ab - 2bc - 2ac$.
15. $a^2 + b^2 + 1 - 2a - 2ab + 2b$.

230. In order to find a general method for extracting the square root of a polynomial, let us consider the relation of $a + b$ to its square, $a^2 + 2ab + b^2$.

The first term a of the root is the square root of the first term a^2 .

The second term of the root can be obtained by dividing the second term $2ab$ by the double of a , the so-called trial divisor;

$$\frac{2ab}{2a} = b.$$

$a + b$ is the root if the given expression is a perfect square. In most cases, however, it is not known whether the given expression is a perfect square, and we have then to consider that $2ab + b^2 = b(2a + b)$, i.e. the sum of trial divisor $2a$, and b , multiplied by b must give the last two terms of the square.

The work may be arranged as follows:

$$\begin{array}{r} a^2 + 2ab + b^2 \quad | \quad a + b \\ \hline a^2 \\ \hline 2a + b \quad | \quad 2ab + b^2 \\ \hline \quad | \quad 2ab + b^2 \end{array}$$

Ex. 1. Extract the square root of $16x^4 - 24x^2y^3 + 9y^6$.

$$\begin{array}{r} 16x^4 - 24x^2y^3 + 9y^6 \quad | \quad 4x^2 - 3y^3 \\ \hline 16x^4 \\ \hline 8x^2 - 3y^3 \quad | \quad -24x^2y^3 + 9y^6 \\ \hline \quad | \quad -24x^2y^3 + 9y^6 \end{array}$$

Explanation. Arrange the expression according to descending powers of x . The square root of $16x^4$ is $4x^2$, the first term of the root. Subtracting the square of $4x^2$ from the trinomial gives the remainder $-24x^2y^3 + 9y^6$. By doubling $4x^2$, we obtain $8x^2$, the trial divisor. Dividing the first term of the remainder, $-24x^2y^3$, by the trial divisor $8x^2$, we obtain the next term of the root $-3y^3$, which has to be added to the trial divisor. Multiply the complete divisor $8x^2 - 3y^3$ by $-3y^3$, and subtract the product from the remainder. As there is no remainder, $4x^2 - 3y^3$ is the required square root.

231. The process of the preceding article can be extended to polynomials of more than three terms. We find the first two terms of the root by the method used in Ex. 1, and consider their sum one term, the first term of the answer. Hence the double of this term is the new trial divisor; by division we find the next term of the root, and so forth.

Ex. 2. Extract the square root of $16a^4 - 24a^3 + 4 - 12a + 25a^2$.

Arranging according to descending powers of a .

	$4a^2 - 3a + 2$
$16a^4 - 24a^3 + 25a^2 - 12a + 4$	
$16a^4$	
First remainder.	$-24a^3 + 25a^2 - 12a + 4$
First trial divisor, $8a^2$.	
First complete divisor, $8a^2 - 3a$.	$-24a^3 + 9a^2$
Second remainder.	$+16a^2 - 12a + 4$
Second trial divisor, $8a^2 - 6a$.	
Second complete divisor, $8a^2 - 6a + 2$.	$+16a^2 - 12a + 4$

As there is no remainder, the required root is $\pm(4a^2 - 3a + 2)$.

232. When some terms of a polynomial are fractional, the one which has the greater power in the denominator is considered the lower power, thus $3x^2 - 5 + \frac{6}{x^2} - 2x + \frac{3}{x}$ arranged according to descending powers of x equals,

$$3x^2 - 2x - 5 + \frac{3}{x} + \frac{6}{x^2}.$$

Ex. 3. Extract the square root of $4x^2 + 25 - \frac{24}{x} - 12x + \frac{16}{x^2}$.

Arranging according to descending powers of x .

	$2x - 3 + \frac{4}{x}$
$4x^2 - 12x + 25 - \frac{24}{x} + \frac{16}{x^2}$	
$4x^2$	
$4x - 3$	$-12x + 25$
	$-12x + 9$
$4x - 6 + \frac{4}{x}$	$+16 - \frac{24}{x} + \frac{16}{x^2}$
	$+16 - \frac{24}{x} + \frac{16}{x^2}$

EXERCISE 90

Extract the square roots of the following expressions:

1. $9 r^4 - 24 r^2 s + 16 s^2$.
2. $a^4 + 13 a^2 - 12 a - 6 a^3 + 4$.
3. $3 a^2 + a^4 - 2 a - 2 a^3 + 1$.
4. $9 x^4 - 12 x^3 + 34 x^2 - 20 x + 25$.
5. $49 a^4 - 42 a^3 b + 37 a^2 b^2 - 12 a b^3 + 4 b^4$.
6. $2 a b - 2 a c - 2 b c + a^2 + b^2 + c^2$.
7. $9 a^2 + 24 a^3 + 46 a^4 + 40 a^5 + 25 a^6$.
8. $9 a^2 - 24 a^3 - 14 a^4 + 40 a^5 + 25 a^6$.
9. $16 x^2 + 40 x^3 + 73 x^4 + 60 x^5 + 36 x^6$.
10. $25 a^2 + 30 a^3 + 69 a^4 + 36 a^5 + 36 a^6$.
11. $49 a^2 + 42 a b + 9 b^2 - 56 a c + 16 c^2 - 24 b c$.
12. $49 x^2 + 56 x y - 70 x z - 40 y z + 16 y^2 + 25 z^2$.
13. $16 a^2 - 40 a b + 24 a c - 30 b c + 25 b^2 + 9 c^2$.
14. $16 x^2 y^2 + 40 x^3 y^3 + 36 x^6 y^6 + 60 x^5 y^5 + 73 x^4 y^4$.
15. $a^{14} x^6 - 6 a^{12} x^7 + 27 a^{10} x^8 - 54 a^8 x^9 + 81 a^6 x^{10}$.
16. $9 x^2 - 54 x^3 + 1 - 6 x + 729 x^6 + 162 x^4$.
17. $1 + 4 a + 20 a^4 - 16 a^5 + 16 a^6$.
18. $a^8 - 4 a^7 + 10 a^6 - 20 a^5 + 25 a^4 - 24 a^3 + 16 a^2$.
19. $25 x^4 + 10 x^2 + 1 + 4 x + 20 x^3 + 24 x^5 + 16 x^6$.
20. $a^2 b^4 c^2 + 8 a^3 b^3 c^2 + 4 a^2 b^2 c^2 - 4 a^2 b^3 c^2 + 4 a^4 b^4 c^2 - 4 a^3 b^4 c^2$.
21. $40 a^9 + 25 a^{10} + 4 a^4 + 12 a^5 + 25 a^6 + 44 a^7 + 46 a^8$.
22. $36 x^2 - 84 x^3 - 47 x^4 + 220 x^5 - 62 x^6 - 144 x^7 + 81 x^8$.
23. $\frac{a^2}{9} + \frac{4 a b}{15} + \frac{4 b^2}{25} + \frac{a c}{2} + \frac{3 b c}{5} + \frac{9 c^2}{16}$.

$$24. \frac{a^2}{9} + \frac{ab}{6} + \frac{2ac}{15} + \frac{b^2}{16} + \frac{bc}{10} + \frac{c^2}{25}.$$

$$25. \frac{1}{9} - \frac{a}{3} + \frac{9a^4}{16} - \frac{3a^3}{4} + \frac{3a^2}{4}.$$

$$26. \frac{4a^2}{9} + ab + \frac{9b^2}{16} - \frac{16ac}{15} - \frac{6bc}{5} + \frac{16c^2}{25}.$$

$$27. \frac{4a^2}{9} - a^3 + \frac{241a^4}{144} - \frac{5a^5}{4} + \frac{25a^6}{36}.$$

$$28. 4 - \frac{12}{x} + \frac{25}{x^2} - \frac{24}{x^3} + \frac{16}{x^4}.$$

$$29. 1 + \frac{4}{a} + \frac{10}{a^2} + \frac{20}{a^3} + \frac{25}{a^4} + \frac{24}{a^5} + \frac{16}{a^6}.$$

$$30. \frac{9a^4}{25b^2} - \frac{3a^2}{5b} + \frac{25}{4} - \frac{5b}{a^2} + \frac{25b^2}{a^4}.$$

$$31. \frac{25}{x^2} + \frac{30}{x} + 69 + 36x + 36x^2.$$

Find to three terms the approximate square roots of :

$$32. 1 + x.$$

$$33. 1 - x$$

$$34. 1 + 2a.$$

$$35. 9 + 4n.$$

233. The square root of arithmetical numbers can be found by a method very similar to the one used for algebraic expressions.

Since the square root of 100 is 10 ; of 10,000 is 100 ; of 1,000,000 is 1000, etc., the integral part of the square root of a number less than 100 has one figure, of a number between 100 and 10,000, two figures, etc. Hence if we divide the digits of the number into groups, beginning at the units, and each group contains two digits (except the last which may contain one or two), then the number of groups is equal to the number of digits in the square root, and the square root of the greatest square in the first group is the first digit in the root. Thus the square root of 96'04' consists of two digits, the first of which is 9 ; the square root of 21'06'81 has three digits, the first of which is 4.

Ex. 1. Find the square root of 7744.

From the preceding explanation it follows that the root has two digits, the first of which is 8. Hence the root is 80 plus an unknown number, and we may apply the method used in algebraic process.

A comparison of the algebraical and arithmetical method given below will show the identity of the methods.

$$\begin{array}{rcl}
 a^2 + 2ab + b^2 \mid \underline{a + b} & & 7744 \mid \underline{80 + 8} \\
 a^2 & & 6400 \\
 2a + b \mid \begin{array}{r} 2ab + b^2 \\ 2ab + b^2 \end{array} & 160 + 8 = 168 & \begin{array}{r} 1344 \\ 1344 \end{array}
 \end{array}$$

Explanation. Since $a = 80$, $a^2 = 6400$, and the first remainder is 1344. The trial divisor $2a = 160$. Therefore $b = 8$, and the complete divisor is 168.

As $8 \times 168 = 1344$, the square root of 7744 equals 88.

Ex. 2. Find the square root of 524,176.

$$\begin{array}{rcl}
 & & a \quad b \quad c \\
 & & 52'41'76 \mid \underline{700 + 20 + 4} \\
 a^2 = & & 49\ 00\ 00 \\
 2a + b = 1400 + 20 = 1420 & & \begin{array}{r} 3\ 41\ 76 \\ 2\ 84\ 00 \end{array} \\
 2(a + b) + c = 1440 + 4 = 1444 & & \begin{array}{r} 57\ 76 \\ 57\ 76 \end{array}
 \end{array}$$

234. In marking off groups in a number which has decimal places, we must begin at the decimal point, and if the right-hand group contains only one digit, annex a cipher.

Thus the groups in .0961 are '.09'61. The group of 16724.1 are 1'67'24.10.

Ex. 3. Find the square root of 6.7 to three decimal places.

$$\begin{array}{rcl}
 6'.70 & \mid & \underline{2.588} \\
 4 & & \\
 45 \mid \begin{array}{r} 2\ 70 \\ 2\ 25 \end{array} & & \\
 508 \mid \begin{array}{r} 4500 \\ 4064 \end{array} & & \\
 5168 \mid \begin{array}{r} 43600 \\ 41344 \end{array} & & \\
 & & 2256
 \end{array}$$

235. Roots of common fractions are extracted either by dividing the root of the numerator by the root of the denominator, or by transforming the common fraction into a decimal.

$$\sqrt{\frac{4}{9}} = \pm \frac{2}{3}; \quad \sqrt{\frac{2}{5}} = \sqrt{\frac{4}{10}}.$$

EXERCISE 91

Extract the square root of:

- | | | |
|------------|--------------|--------------------------|
| 1. 1024. | 9. 27,889. | 18. 3,294,225. |
| 2. 5625. | 10. 75,076. | 19. 10,227,204. |
| 3. 4761. | 11. 57,121. | 20. 17,850,625. |
| 4. 8836. | 12. 772.84. | 21. 33,790,969. |
| 5. 1369. | 13. 328,329. | 22. $30\frac{1}{4}$. |
| 6. 6724. | 14. 390,625. | 23. $\frac{961}{2304}$. |
| 7. 9604. | 15. 74.1321. | 24. $1\frac{9}{16}$. |
| 8. 15,129. | 16. 6037.29. | 25. $\frac{1}{66564}$. |
| | 17. 857,476. | |

Find to three decimal places the square roots of the following numbers:

- | | | | |
|---------|------------|-----------------------|---------------------|
| 26. 8. | 28. 11.5. | 30. $3\frac{1}{10}$. | 32. $\frac{2}{3}$. |
| 27. 10. | 29. 147.6. | 31. $4\frac{1}{4}$. | 33. $\frac{5}{6}$. |

236. The **fourth root** of an algebraic expression or arithmetical number is obtained by extracting the square root of the square root of the given quantity. Similarly the eighth root may be obtained by three successive extractions of the square root.

EXERCISE 92

Extract the fourth root of:

- $a^8 - 4a^6 + 6a^4 - 4a^2 + 1.$
- $m^4 + 8m^3y + 24m^2y^2 + 32my^3 + 16y^4.$
- $81x^3 + 108x^6y^2 + 54x^4y^4 + 12x^4y^6 + y^8.$

$$4. 16 a^8 - 96 a^6 b^3 + 216 a^4 b^6 - 216 a^2 b^9 + 81 b^{12}.$$

$$5. 279,841.$$

$$7. 614,656.$$

$$6. 456,976.$$

$$8. 1,874,161.$$

(For Cube Roots see Appendix IV.)

REVIEW EXERCISE IV

$$1. \text{ Expand } (3a - b + c)^3.$$

$$2. \text{ Find the value of } \frac{\frac{7}{12} + \frac{11}{18} - \frac{1}{36}}{\frac{7}{12} - \frac{11}{18} + \frac{1}{36}}.$$

$$3. \text{ Simplify } \frac{p^2 - q^3}{6} + \frac{q^3 - p^2}{7} - \frac{p^2 - q^3}{42}.$$

$$4. \text{ Find the mean proportional between } \frac{a^2 - b^2}{a^2 + b^2} \text{ and } a^4 - b^4.$$

$$5. \text{ Find the third proportional to } \frac{(a^2 - b^2)^2}{(a + b)^4} \text{ and } \frac{a - b}{a + b}.$$

$$6. \text{ Simplify } [(a - b)^5 + (a + b)^5][(a - b)^5 - (a + b)^5].$$

Solve the following equations:

$$7. \begin{cases} y : z = 1 : 2, \\ x : z = 14 : 98, \\ x : 7 = 12 : 42. \end{cases}$$

$$8. \begin{cases} x - y + z - u = -2, \\ 2x + y - 3z + u = -1, \\ 4x - 2y - z - u = -7, \\ 5x + 2y + 2z + u = 19. \end{cases}$$

$$9. \begin{cases} \frac{x}{4} - \frac{y - 1}{8} = 1\frac{3}{8}, \\ \frac{x}{3} - \frac{y - 2x}{6} = \frac{y + 16}{6}. \end{cases}$$

$$10. \begin{cases} x - 2y - 3z + u + v = -2, \\ 2x - 3u + 2z - 4v + y = -22, \\ 2y - 6z - 2u + 3x - v = -17, \\ 5z - 2v + 3u - 4x - y = 10, \\ 2v - 4u + 3y - z + x = -4. \end{cases}$$

$$11. \begin{cases} a^x \cdot a^y = a^{100}, \\ a^x \div a^y = a^{80}. \end{cases}$$

12. Find the numerical value of

$$3^2 - 14 \cdot 3^3 - 13 \cdot 3^2 - 11 \cdot 3^3 + 25 \cdot 3^3 + 15 \cdot 3^2.$$

13. Find the value of $(5000 + 7)^3$.

14. Find the 9th root of $2^{36} a^{45} b^9 \frac{(a+b)^{27}}{a^9}$.

Extract the square root of the following expressions:

$$15. \frac{4}{9} a^2 - ab + \frac{10}{9} ac + \frac{9}{16} b^2 - \frac{5}{4} bc + \frac{25}{36} c^2.$$

$$16. \frac{4a^4 - 28a^2b^2 + 49b^4}{x^4 + 10x^2z + 25z^2}.$$

$$17. \frac{m^2 + 2m + 1}{a^2} + \frac{2(m^2 + 3m + 2)}{a^3} + \frac{3m^2 + 12m + 10}{a^4} \\ + \frac{2(m^2 + 5m + 6)}{a^5} + \frac{m^2 + 6m + 9}{a^6}.$$

18. Solve the equations:

$$(a) \quad (-2)^x = -32.$$

$$(b) \quad (-2)^x = 64.$$

$$(c) \quad (-3)^x = -27.$$

19. Solve the following system:

$$\begin{cases} x : y : z : u = 2 : 3 : 4 : 5, \\ 3x - 2y + 4z - 3u = 1. \end{cases}$$

20. Reduce to lowest terms $\frac{2x^3 + x^2 - 8x + 5}{7x^2 - 12x + 5}$.

21. Reduce to lowest terms $\frac{a^2 + b^2 + c^2 + 2ab + 2ac + 2bc}{a^2 - b^2 - c^2 - 2bc}$.

22. Simplify $(\sqrt{(a^{2m})})^{3n}$.

Solve the equations:

23. $\frac{3x - 5}{4} : \left(\frac{5x + 6}{3} - 1 \right) = 2 : 7$.

24. $\frac{x^p - 2x^{p-1}}{2x - 4} + \frac{x^p}{x - 2} = \frac{3x^p}{2x - 2}$.

CHAPTER XIV

THE THEORY OF EXPONENTS

237. The following four fundamental laws for positive integral exponents have been developed in preceding chapters :

I. $a^m \cdot a^n = a^{m+n}.$

II. $a^m \div a^n = a^{m-n},$ provided $m > n.*$

III. $(a^m)^n = a^{mn}.$

IV. $(ab)^m = a^m \cdot b^m.$

The first of these laws is the direct consequence of the definition of power, while the second and third are consequences of the first.

FRACTIONAL AND NEGATIVE EXPONENTS

238. Fractional and negative exponents, such as $2^{\frac{1}{3}}, 4^{-3},$ have no meaning according to the original definition of power, and we may choose for such symbols any definition that is convenient for other work.

It is, however, very important that all exponents should be governed by the same laws; hence, instead of giving a formal definition of fractional and negative exponents, we let these quantities be what they must be if the exponent law of multiplication is generally true.

239. We assume, therefore, that $a^m \cdot a^n = a^{m+n},$ for all values of m and $n.$ Then the law of involution, $(a^m)^n = a^{mn},$ must be

* The symbol $>$ means "is greater than"; similarly $<$ means "is smaller than."

true for positive integral values of n , since the raising to a positive integral power is only a repeated multiplication.

Assuming these two laws, we try to discover the meaning of $8^{\frac{1}{3}}$, a^0 , 4^{-2} , $a^{\frac{m}{n}}$, etc. In every case we let the unknown quantity equal x , and apply to both members of the equation that operation which makes the negative, fractional, or zero exponent disappear.

240. To find the meaning of a fractional exponent ; e.g. $a^{\frac{1}{3}}$.

Let $x = a^{\frac{1}{3}}$.

The operation which makes the fractional exponent disappear is evidently the raising of both members to the third power.

Hence $x^3 = (a^{\frac{1}{3}})^3$.

Or $x^3 = a$.

Therefore $x = \sqrt[3]{a}$.

Similarly, to find a meaning for $a^{\frac{p}{q}}$,

we let $x = a^{\frac{p}{q}}$.

Raising both members to the q th power, $x^q = a^p$.

Taking the q th root of both members, $x = \sqrt[q]{a^p}$,

or $a^{\frac{p}{q}} = \sqrt[q]{a^p}$.

Hence we define $a^{\frac{p}{q}}$ to be the q th root of a^p .

EXERCISE 93

Find the values of :

- | | | | |
|---|---|----------------------------|--------------------------------------|
| 1. $8^{\frac{1}{3}}$. | 5. $9^{\frac{2}{3}}$. | 9. $16^{\frac{3}{4}}$. | 13. $(-32)^{\frac{1}{5}}$. |
| 2. $4^{\frac{1}{2}}$. | 6. $27^{\frac{2}{3}}$. | 10. $1^{\frac{7}{8}}$. | 14. $(\frac{1}{4})^{\frac{1}{2}}$. |
| 3. $9^{\frac{1}{2}}$. | 7. $32^{\frac{2}{5}}$. | 11. $0^{\frac{2}{3}}$. | 15. $(\frac{1}{27})^{\frac{1}{3}}$. |
| 4. $64^{\frac{1}{3}}$. | 8. $125^{\frac{2}{3}}$. | 12. $(-8)^{\frac{1}{3}}$. | 16. $(\frac{1}{4})^{\frac{3}{2}}$. |
| 17. $(a^2 + 2ab + b^2)^{\frac{1}{2}}$. | 18. $(a^3 - 3a^2 + 3a - 1)^{\frac{1}{3}}$. | | |

Write the following expressions as radicals:

19. $a^{\frac{2}{3}}$.

23. $\left(\frac{1}{a}\right)^{\frac{4}{5}}$.

27. $3^{\frac{3}{p}}$.

20. $x^{\frac{3}{5}}$.

24. $a^{\frac{4}{m}}$.

28. $a^{\frac{m}{n}}$.

21. $m^{1\frac{1}{4}}$.

25. $b^{\frac{m}{4}}$.

29. $a^{\frac{1}{2}}b^{\frac{1}{3}}$.

22. $(ab)^{\frac{4}{3}}$.

26. $2^{\frac{2}{m}}$.

30. $a^{\frac{m+n}{m-n}}$.

Express with fractional exponents:

31. $\sqrt[3]{a^2}$.

35. $\sqrt[3]{3}$.

39. $\sqrt[6]{a} \cdot \sqrt[3]{b}$.

32. $\sqrt[7]{a^4}$.

36. $\sqrt[8]{a^4}$.

40. $\sqrt{a} \cdot \sqrt[3]{b} \cdot \sqrt[4]{c}$.

33. $\sqrt[8]{a^3}$.

37. $\sqrt[12]{x^3}$.

41. $\sqrt[4]{b^2}$.

34. \sqrt{a} .

38. $\sqrt[4]{ab}$.

42. $\sqrt[2a]{a^{a^2+a}}$.

Find the values of:

43. $\sqrt[3]{4}$.

46. $\sqrt[3]{2}$.

49. $\sqrt[1]{2}$.

44. $\sqrt[3]{4}$.

47. $\sqrt[5]{3}$.

50. $\sqrt[0]{3}$.

45. $\sqrt[0]{2}$.

48. $\sqrt[1]{2^{\frac{1}{2}}}$.

241. To find the meaning of zero exponent, *e.g.* a^0 .

Let $x = a^0$.

The operation which makes the zero exponent disappear is evidently a multiplication by any power of a , *e.g.* a^2 .

$$a^2x = a^{0+2} = a^2.$$

$$x = \frac{a^2}{a^2} = 1.$$

or

$$a^0 = 1.$$

Therefore the zero power of any number is equal to unity.

NOTE. If, however, the base is zero, $\frac{a^2}{a^2}$ is indeterminate; hence 0^0 is indeterminate.

242. To find the meaning of a negative exponent, e.g. a^{-n} .

Let $x = a^{-n}$.

Multiplying both numbers by a^n , $a^n x = a^0$.

Or $a^n x = 1$.

Hence $x = \frac{1}{a^n}$.

Therefore $a^{-n} = \frac{1}{a^n}$.

243. Factors may be transferred from the numerator to the denominator of a fraction, or vice versa, by changing the sign of the exponent.

$$1. \quad a^{-n} = \frac{1}{a^n}.$$

$$2. \quad \frac{1}{a^{-n}} = \frac{1}{\frac{1}{a^n}} = a^n.$$

NOTE. The fact that $a^0 = 1$ sometimes appears peculiar to beginners. It loses its singularity if we consider the following equations, in which each is obtained from the preceding one by dividing both numbers by a .

$$a^3 = 1 \cdot a \cdot a \cdot a$$

$$a^2 = 1 \cdot a \cdot a$$

$$a^1 = 1 \cdot a$$

$$a^0 = 1$$

$$\hline a^{-1} = \frac{1}{a}$$

$$a^{-2} = \frac{1}{a^2} \text{ etc.}$$

EXERCISE 94

Find the values of:

1. 4^{-2} .

3. 1^{-9} .

5. 1^{-10} .

7. 3^0 .

2. 2^{-4} .

4. 100^0 .

6. 2^{-3} .

8. $\left(\frac{1}{2}\right)^0$.

- | | | |
|----------------------------|-------------------------------------|--|
| 9. $\frac{1}{2^0}$. | 14. $25^{-\frac{1}{2}}$. | 19. $\sqrt[3]{-10}$. |
| 10. $\frac{1}{2^{-2}}$. | 15. $32^{-\frac{1}{3}}$. | 20. $\sqrt[4]{-2}$. |
| 11. $\frac{1}{3^{-1}}$. | 16. $625^{-\frac{1}{4}}$. | 21. $\sqrt[5]{-32}$. |
| 12. $(\frac{3}{4})^{-2}$. | 17. $\frac{1}{36^{-\frac{3}{2}}}$. | 22. $\sqrt[0]{\frac{1}{2}}$. |
| 13. $8^{-\frac{1}{3}}$. | 18. 0^{-5} . | 23. $8^{\frac{2}{3}} \cdot 4^{-2}$. |
| | | 24. $8^{\frac{1}{3}} \cdot 4^{-\frac{1}{2}}$. |

Express with positive exponents:

- | | | |
|---------------------------------------|------------------------------------|---|
| 25. a^{-4} . | 30. $\frac{-5a^{-2}}{b^3}$. | 33. $\frac{3a^{-5}b^{-5}}{6^{-2}ad^{-4}}$. |
| 26. $2a^{-6}$. | 31. $\frac{5^{-1}a^2}{b^{-3}}$. | 34. $\frac{9a^7b^{-\frac{1}{2}}}{4a^{-\frac{3}{2}}b^{\frac{5}{2}}}$. |
| 27. $5a^{-2}x^5$. | | |
| 28. $7a^{-9}b^{-11}c^5$. | 32. $\frac{6m^{-2}n}{5^{-1}m^4}$. | |
| 29. $\frac{6a^2b^2c^{-1}}{5d^{-4}}$. | | |

Write without denominators:

- | | | |
|------------------------------|-------------------------------------|--|
| 35. $\frac{12a^2b^2}{c^3}$. | 37. $\frac{a^2b^2c^2}{x^9y^9z^9}$. | 39. $\frac{x^{-2}y^{\frac{1}{2}}}{x^{\frac{1}{2}}y}$. |
| 36. $\frac{17a}{b^{12}}$. | 38. $\frac{a^{-1}b}{xy^4}$. | 40. $\frac{1}{6abc}$. |

Write with radical signs and positive exponents:

- | | | |
|-----------------------------|------------------------------------|---|
| 41. $a^{\frac{1}{2}}$. | 47. $2m^{-\frac{3}{n}}$. | 51. $\frac{m^{-\frac{1}{x}}}{3}$. |
| 42. $a^{-\frac{1}{2}}$. | 48. $2^{-\frac{1}{m}}$. | 52. $\frac{1}{a^{-\frac{1}{p}}}$. |
| 43. $5a^{\frac{1}{3}}$. | 49. $\frac{1}{6^{-\frac{2}{p}}}$. | 53. $\frac{a^{\frac{1}{2}}}{2^{-1}b^{-2}}$. |
| 44. $5a^{-\frac{1}{2}}$. | 50. $\frac{2}{a^{-\frac{x}{3}}}$. | 54. $\sqrt[3]{a} \cdot \sqrt[2]{a} \cdot \sqrt[6]{a}$. |
| 45. $(5a)^{-\frac{1}{2}}$. | | |
| 46. $2m^{-\frac{n}{3}}$. | | |

THE LAWS FOR NEGATIVE AND FRACTIONAL EXPONENTS

244. Exponent law of division for any values of m and n .

To prove $a^m \div a^n = a^{m-n}$ for any value of m and n .

$$\frac{a^m}{a^n} = a^m \times \frac{1}{a^n} = a^m \cdot a^{-n} \quad (\S 243)$$

$$= a^{m-n}. \quad (\S 239)$$

Hence the law is true for any values of m and n .

245. Exponent law of involution for any values of m and n .

To prove $(a^m)^n = a^{mn}$ for any values of m and n .

Case 1. Let m have any value, and n be a positive integer. This was proved in a preceding chapter (§ 239).

Case 2. Let m have any value, and n be a positive fraction $\frac{p}{q}$.

$$(a^m)^{\frac{p}{q}} = \sqrt[q]{(a^m)^p} \quad (\S 240)$$

$$= \sqrt[q]{a^{mp}} \quad (\text{Case 1})$$

$$= a^{\frac{mp}{q}}. \quad (\S 240)$$

Case 3. Let m be any number and n be negative.

$$(a^m)^{-n} = \frac{1}{(a^m)^n} = \frac{1}{a^{mn}} = a^{-mn}.$$

246. In a similar manner it can be proved that the law $(ab)^m = a^m b^m$ is true for fractional and negative exponents.

Case 1. Let the exponent be $\frac{p}{q}$ when p and q represent positive integers.

Then
$$(a^{\frac{p}{q}} \cdot b^{\frac{p}{q}})^q = a^p b^p = (ab)^p.$$

Hence
$$a^{\frac{p}{q}} \cdot b^{\frac{p}{q}} = (ab)^{\frac{p}{q}}.$$

Case 2. Let the exponent be $-n$, where n represents any number; then

$$(ab)^{-n} = \frac{1}{(ab)^n} = \frac{1}{a^n b^n} = a^{-n} b^{-n}.$$

Hence the four laws of exponents are true for any value of the exponents, and we have, in general,*

Fractional and negative exponents are treated by the same methods as positive integral exponents.

247. Examples relating to roots can be reduced to examples containing fractional exponents.

$$\begin{aligned}\text{Ex. 1. } (a^{\frac{1}{2}}b^{-\frac{1}{4}})^{\frac{2}{3}} \div (a^{\frac{3}{2}}b^{-\frac{3}{2}})^{\frac{1}{3}} &= a^{\frac{1}{3}}b^{-\frac{1}{6}} \div a^{\frac{1}{2}}b^{-\frac{1}{2}} = a^{-\frac{1}{6}}b^{\frac{1}{3}} \\ &= \frac{b^{\frac{1}{3}}}{a^{\frac{1}{6}}} = \sqrt[6]{\frac{b^2}{a}}.\end{aligned}$$

$$\text{Ex. 2. } \left(\frac{2\sqrt{a}\sqrt[3]{b}}{3\sqrt[6]{a}\sqrt[2]{b}}\right)^6 = \left(\frac{2a^{\frac{1}{2}}b^{\frac{1}{3}}}{3a^{\frac{1}{3}}b^{\frac{1}{2}}}\right)^6 = \frac{64a^3b^2}{729ab^3} = \frac{64a^2}{729b}.$$

248. Expressions containing radicals should be simplified as follows:

- (a) Write all radical signs as fractional exponents.
- (b) Perform the operation indicated.
- (c) Remove the negative exponents.
- (d) If required, remove the fractional exponents.

NOTE. *Negative exponents should not be removed until all operations of multiplication, division, etc., are performed.*

EXERCISE 95

- | | |
|--|---|
| 1. $a^5 \cdot 2a^{-4} \cdot 3a^2 \cdot 2a^{-1}$. | 7. $a^5 \div a^{-4}$. |
| 2. $6a^3 \cdot 2a^{-4}$. | 8. $12a^{-4} \div 6a^{-5}$. |
| 3. $x^{\frac{1}{2}} \cdot x^{\frac{1}{3}} \cdot x^{\frac{1}{6}}$. | 9. $14a^{-\frac{7}{2}} \div 2a^{-\frac{5}{2}}$. |
| 4. $5^{\frac{1}{4}} \cdot 5^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} \cdot 5^{\frac{1}{4}}$. | 10. $4m^{\frac{n}{2}} \cdot 3m^{-\frac{n}{3}} \div 6m^{-n}$. |
| 5. $x^{\frac{2}{3}} \cdot 2x^{-\frac{1}{3}}$. | 11. $(\sqrt[4]{3})^8$. |
| 6. $2a^{-\frac{7}{2}} \cdot 2a^{-\frac{5}{2}} \cdot 2a^4$. | 12. $\sqrt{2} \cdot \sqrt[3]{2} \cdot \sqrt[6]{2}$. |

* Irrational and imaginary exponents cannot be considered at this stage of the work.

13. $\sqrt[6]{2^5} \div \sqrt[3]{2^2}$.

14. $\sqrt[5]{7^6} \cdot \sqrt[5]{7^4}$.

15. $\sqrt[3]{3} \cdot \sqrt[6]{3} \div \sqrt[2]{3}$.

16. $2 \div \sqrt{2}$.

17. $2 \cdot \sqrt[4]{2} \div \sqrt[4]{2^3}$.

18. $\sqrt[3]{7} \cdot \sqrt[4]{7^2} \cdot \sqrt[6]{7}$.

19. $\sqrt[4]{4} \cdot \sqrt[6]{8}$.

HINT, $4 = 2^2$.

20. $\sqrt[5]{4} \cdot \sqrt[15]{512}$.

21. $\sqrt[6]{81} \cdot \sqrt[9]{27}$.

22. $\sqrt[8]{32} \cdot \sqrt[12]{128}$.

23. $(x^{\frac{2}{3}})^{\frac{3}{4}}$.

24. $(\sqrt[5]{x^4})^{\frac{5}{6}}$.

25. $(\sqrt[3]{p^2})^{\frac{6}{7}}$.

26. $\sqrt[3]{x} \cdot \sqrt[4]{x} \cdot \sqrt[6]{x}$.

40. $(\sqrt{a})^{\frac{1}{3}}$.

41. $\sqrt[3]{a^4}$.

42. $\sqrt[4]{\sqrt[5]{a}}$.

43. $\sqrt[3]{\sqrt[3]{a}}$.

50. $\sqrt[4]{\frac{x^2 y^2}{a^6 b^4}}$.

51. $\left(\frac{81 x^{-8} y^6}{25 a^4 b^{-10}}\right)^{-\frac{3}{2}}$.

52. $\left(\frac{(-3)^{\frac{1}{3}} \cdot \sqrt[6]{x^{-5}}}{(\frac{4}{5})^{-\frac{1}{3}} \cdot \sqrt[9]{y^{-4}}}\right)^{-\frac{9}{4}}$.

53. $\left(\frac{24 \sqrt[4]{b^3} \cdot \sqrt{a^3}}{16 \sqrt[4]{a^3} \cdot \sqrt{a^5}}\right)^{\frac{4}{3}}$.

27. $\sqrt[4]{a^3} \cdot \sqrt[6]{a} \cdot \sqrt[8]{a^5}$.

28. $\sqrt[5]{a^4} \cdot \sqrt[10]{a^3} \cdot \sqrt[15]{a^7}$.

29. $\sqrt[12]{a^7} \cdot \sqrt{a} \cdot \sqrt[3]{a^2} \cdot \sqrt[18]{a^{11}}$.

30. $\sqrt[4]{125} \cdot \sqrt[6]{5} \cdot \sqrt[3]{25} \cdot \sqrt[18]{5}$.

31. $\sqrt[4]{a^3} \div \sqrt[3]{a^2}$.

32. $\sqrt[6]{x^5} \div \sqrt[4]{x^3}$.

33. $\sqrt[8]{a^7} \div \sqrt[12]{a^5}$.

34. $\sqrt[6]{a^{5x}} \div \sqrt[9]{a^{4x}}$.

35. $\sqrt[6]{a^{n+2}} \div \sqrt[12]{a^{n+1}}$.

36. $\sqrt[3]{a} \cdot \sqrt[4]{a} \div \sqrt[6]{a}$.

37. $\sqrt[4]{x} \div (\sqrt[6]{x^5} \cdot \sqrt[9]{x^4})$.

38. $\frac{\sqrt{2} \cdot \sqrt[3]{4}}{\sqrt[6]{32}}$.

39. $\frac{a^{+\frac{1}{2}} \sqrt{x^a} \cdot a^{-\frac{1}{2}} \sqrt{x^b}}{a^{2-b^2} \sqrt{x^2}}$.

44. $\sqrt[8]{\sqrt[5]{a^{12}}}$.

45. $\sqrt[9]{\sqrt[5]{a^6}}$.

46. $(\sqrt{x^3 y^4})^5$.

47. $(\sqrt{x^2 y^{-3}})^{-2}$.

48. $\left(\frac{27 a^{-3}}{64 b^{-3}}\right)^{-\frac{1}{3}}$.

49. $\sqrt{\left(\frac{a^2 b}{3 a^4 x^2}\right)^3}$.

54. $\frac{\sqrt[3]{ax^2}}{\sqrt[4]{a^2x}} \cdot \frac{\sqrt[4]{ax^2}}{\sqrt[3]{a^2x}}$.

55. $\sqrt[6]{(\sqrt{a})^{-1\frac{1}{2}} \cdot (\sqrt[3]{\sqrt{a}})^{-3}}$.

56. $\sqrt[6]{(a^{\frac{1}{2}} \cdot b^{\frac{1}{3}})^{-\frac{5}{2}}}$.

57. $(\sqrt[2]{a^n} \cdot \sqrt[4]{a^{3n}})^{-\frac{1}{5}}$.

58. $\left\{ \frac{\sqrt{a}}{\sqrt[5]{b^{-1}}} \cdot \left(\frac{b}{a}\right)^{\frac{1}{2}} \cdot \frac{\sqrt[6]{a}}{b^{-1}} \right\}^2$.

$$59. \left(\frac{x^{-3}y}{x^2y^{-3}} \right)^{-3} \cdot \left(\frac{x^{-2}y^{-1}}{x^3y^3} \right)^5.$$

$$63. \frac{3^{n+1}}{4^{n+1}} \div \frac{(3^2)^{n-1}}{3^{n^2-1}}.$$

$$60. \sqrt[7]{\left(\frac{1}{1 \div \sqrt{a^7}} \right)^2}.$$

$$64. \left\{ \frac{a^{m+n}}{\sqrt[m]{a^{m^2-mn}}} \cdot a^{-m} \right\}^{\frac{1}{m}}.$$

$$61. \sqrt[m]{x^{-2}y^{-3}z^{-1}} \div \sqrt[2m]{x^{-4}y^{6-2m}}.$$

$$65. \sqrt[n]{\frac{(a+b)^6}{(a-b)^{-4}}} \div \sqrt[2n]{\frac{(a+b)^3}{(a-b)^{-3}}}.$$

$$62. \frac{\sqrt{x^4} \cdot \sqrt[4]{x^2} \cdot \sqrt{x^5}}{\sqrt[3]{x} \cdot x^{\frac{3}{4}}} \div (x^{\frac{3}{4}})^{\frac{2}{3}}.$$

$$66. \sqrt{\frac{a}{\sqrt[3]{x}}} \cdot \sqrt{\frac{b}{\sqrt[3]{x}}} \cdot \sqrt{\frac{c}{\sqrt[3]{x}}}.$$

249. If we wish to arrange terms according to descending powers of x , we have to remember that the term which does not contain x may be considered as a term containing x^0 . The powers of x arranged are:

$$x^3, x^2, x, x^0, x^{-1}, x^{-2}, x^{-3}.$$

Ex. 1. Multiply $3x^{-1} + x - 5$ by $2x - 1$.

Arrange in descending powers of x .

Check. If $x = 1$

$$\begin{array}{r} x - 5 + 3x^{-1} \\ 2x - 1 \\ \hline 2x^2 - 10x + 6 \\ - \quad x + 5 - 3x^{-1} \\ \hline 2x^2 - 11x + 11 - 3x^{-1} \end{array} \quad \begin{array}{l} = -1 \\ = +1 \\ \\ \\ = -1 \end{array}$$

Ex. 2. Divide

$$\sqrt[3]{a^4} - 4\sqrt[3]{b^4} - 6a\sqrt[3]{b} + 9\sqrt[3]{a^2b^2} \text{ by } 2\sqrt[3]{b^2} - 3\sqrt[3]{ab} + \sqrt[3]{a^2}.$$

$$\begin{array}{r} a^{\frac{4}{3}} - 6ab^{\frac{1}{3}} + 9a^{\frac{2}{3}}b^{\frac{2}{3}} - 4b^{\frac{4}{3}} \quad \left| \begin{array}{l} a^{\frac{2}{3}} - 3a^{\frac{1}{3}}b^{\frac{1}{3}} + 2b^{\frac{2}{3}} \\ a^{\frac{2}{3}} - 3a^{\frac{1}{3}}b^{\frac{1}{3}} + 2b^{\frac{2}{3}} \\ \hline -3ab^{\frac{1}{3}} + 7a^{\frac{2}{3}}b^{\frac{2}{3}} - 4b^{\frac{4}{3}} \\ -3ab^{\frac{1}{3}} + 9a^{\frac{2}{3}}b^{\frac{2}{3}} - 6a^{\frac{1}{3}}b \\ \hline -2a^{\frac{2}{3}}b^{\frac{2}{3}} + 6a^{\frac{1}{3}}b - 4b^{\frac{4}{3}} \\ -2a^{\frac{2}{3}}b^{\frac{2}{3}} + 6a^{\frac{1}{3}}b - 4b^{\frac{4}{3}} \end{array} \right. \end{array}$$

Ex. 3. Find the square root of

$$x^2 - 2x\sqrt{a} + 3a - \frac{2a\sqrt{a}}{x} + \frac{a^2}{x^2}.$$

Write with fractional and negative exponents:

$$\begin{array}{r} x^2 - 2a^{\frac{1}{2}}x + 3a - 2a^{\frac{3}{2}}x^{-1} + a^2x^{-2} \quad | \quad x - a^{\frac{1}{2}} + ax^{-1} \\ \hline 2x - a^{\frac{1}{2}} \quad | \quad -2a^{\frac{1}{2}}x + 3a \\ \quad \quad \quad | \quad -2a^{\frac{1}{2}}x + a \\ \hline 2x - 2a^{\frac{1}{2}} + ax^{-1} \quad | \quad 2a - 2a^{\frac{3}{2}}x^{-1} + a^2x^{-2} \\ \quad \quad \quad | \quad 2a - 2a^{\frac{3}{2}}x^{-1} + a^2x^{-2} \end{array}$$

This answer may be written without negative and fractional exponents:

$$x - \sqrt{a} + \frac{a}{x}.$$

EXERCISE 96

Perform the operations indicated:

1. $(3a^{-5} - 4a^{-4} + 5a^{-3})(2a^{-2} - a^{-1}).$
2. $(2x^2 - 3x^{-1} + 2x - 3)(2x - 1).$
3. $(x^{-m} + x^m + 1)(x^{-m} + x^m - 1).$
4. $(3a + 5\sqrt{a} + 3)(3a - 5\sqrt{a} + 3).$
5. $(a^{\frac{3}{2}} + 2a - a^{\frac{1}{2}} - 2)(2a - 1).$
6. $(\sqrt{a} + \sqrt{b} + 1)(\sqrt{a} + \sqrt{b} - 1).$
7. $(\sqrt[3]{a^2} + 2\sqrt[3]{ab} - \sqrt[3]{b^2})(\sqrt[3]{a^2} - 3\sqrt[3]{ab} + 2\sqrt[3]{b^2}).$
8. $(a^m - 2 + a^{-m} - a^{-2m})(a^m + a^{-m} + 1).$
9. $(42x - 10x^{\frac{1}{2}}y^{\frac{1}{2}} - 12y) \div (7x^{\frac{1}{2}} + 3y^{\frac{1}{2}}).$
10. $(2a^{\frac{3}{2}} + 18 - 3a^{\frac{1}{2}} - 7a) \div (2a^{\frac{1}{2}} + 3).$
11. $(4a^2 + 30a^{-1} - 9 - 25a^{-2}) \div (2 + 3a^{-1} - 5a^{-2}).$
12. $(12\sqrt[3]{a^2} - 28\sqrt[3]{a} + 15) \div (6\sqrt[3]{a} - 5).$
13. $(a - b) \div (\sqrt[3]{a} - \sqrt[3]{b}).$

$$\left\{ \begin{array}{l} 14. (4\sqrt[3]{a} + 3 + \sqrt[3]{a^4}) \div (\sqrt[3]{a^2} + 3 - 2\sqrt[3]{a}). \\ 15. (x^{2m} - 6x^m - 19 + 84x^{-m}) \div (1 - 7x^{-m}). \\ 16. (a + b + 2\sqrt[3]{ab^2} + 2\sqrt[3]{a^2b}) \div (\sqrt[3]{a^2} + \sqrt[3]{b^2} + \sqrt[3]{ab}). \end{array} \right.$$

$$\left\{ \begin{array}{l} 17. \sqrt{9a^{-2} + 24a^{-1} + 46 + 40a + 25a^2}. \end{array} \right.$$

$$18. \sqrt{x^2 - 6x^{\frac{3}{2}} + 13x - 12x^{\frac{1}{2}} + 4}.$$

$$19. \sqrt{x^{-2} - 2x + x^2 - 2x^{-1} + 3}.$$

$$20. \sqrt{9x^2 - 12\sqrt{x^3} + 34x - 20\sqrt{x} + 25}.$$

$$21. (49a^2 - 42\sqrt{a^3b} + 37ab - 12\sqrt{ab^3} + 4b^2)^{\frac{1}{2}}.$$

$$22. (2x^m + 2x^{-m} + 3 + x^{2m} + x^{-2m})^{\frac{1}{2}}.$$

$$23. (2ax + 16\sqrt[3]{9a^4x^4} - 8\sqrt[6]{72a^7x^7})^{\frac{1}{2}}.$$

$$24. (1 + a + 2\sqrt{a} + \sqrt[3]{a^4} + 2\sqrt[3]{a^2} + 2\sqrt[6]{a^7})^{\frac{1}{2}}.$$

$$25. \left(\frac{4a}{9\sqrt[3]{b^2}} + \frac{2\sqrt{a}}{\sqrt[3]{b}} + \frac{19}{12} - \frac{3\sqrt[3]{b}}{2\sqrt{a}} + \frac{\sqrt[3]{b^2}}{4a} \right)^{\frac{1}{2}}.$$

$$26. (3 + \sqrt{5} + 2\sqrt{6})(\sqrt{5} + \sqrt{6}).$$

$$27. (2 + 3\sqrt{3} + 4\sqrt{6})(2\sqrt{3} + 3\sqrt{6}).$$

$$28. (4\sqrt{5} + 5\sqrt{6} + 4)(5 + 6\sqrt{6}).$$

$$29. \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{c}} \right) \left(\sqrt{\frac{b}{a}} + \sqrt{\frac{c}{b}} \right).$$

$$30. (\sqrt{a} + \sqrt{b} + 1) \left(\sqrt{\frac{1}{a}} + \sqrt{\frac{1}{b}} - 1 \right).$$

$$31. (2 - \sqrt{x} + \frac{1}{2}x)^2.$$

$$32. (1 - \sqrt{a} + a)^2.$$

$$\times 33. (1 - 4\sqrt{a} + 6a - 4a\sqrt{a} + a^2)^{\frac{1}{2}}.$$

$$\times 34. \left(\frac{4\sqrt{a}}{3b} - \frac{4}{3} - \frac{415b}{36\sqrt{a}} + \frac{161b^2}{18a} - \frac{2b^3}{\sqrt{a^3}} \right) \div \left(\frac{2\sqrt{a}}{b} - \frac{3}{2} + \frac{b}{3\sqrt{a}} \right).$$

Find by inspection the values of the following:

- | | |
|--|--|
| 35. $(x^{\frac{1}{2}} + 3)(x^{\frac{1}{2}} - 5)$. | 49. $(a + 1 + a^{-1})^2$. |
| 36. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} - y^{\frac{1}{2}})$. | 50. $(m^a + 2 - m^{-a})^2$. |
| 37. $(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2$. | 51. $(x^{-1} + y^{-1} + z^{-1})^2$. |
| 38. $(a + a^{-1})^2$. | 52. $(x^{-2} - y^{-2})(x^{-1} - y^{-1})$. |
| 39. $(x - x^{-1})^2$. | 53. $(a - b) \div (a^{\frac{1}{2}} + b^{\frac{1}{2}})$. |
| 40. $(x^{-\frac{1}{2}} - 9)(x^{-\frac{1}{2}} + 9)$. | 54. $(a - b) \div (a^{\frac{1}{3}} - b^{\frac{1}{3}})$. |
| 41. $(x^{\frac{1}{2}} + 1)^3$. | 55. $(a^{-3} - b^{-3}) \div (a^{-1} - b^{-1})$. |
| 42. $(a^{\frac{1}{3}} + 1)^2$. | 56. $(x + 8) \div (x^{\frac{1}{3}} + 2)$. |
| 43. $(a^{\frac{1}{3}} + b^{\frac{1}{3}})^2$. | 57. $\sqrt{m + 2m^{\frac{1}{2}} + 1}$. |
| 44. $(a^{\frac{1}{3}}b^{\frac{1}{3}} - a^{\frac{1}{2}}b^{\frac{1}{2}})(a^{\frac{1}{3}}b^{\frac{1}{3}} + a^{\frac{1}{2}}b^{\frac{1}{2}})$. | 58. $\sqrt{a^{\frac{2}{3}} + 2a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}}$. |
| 45. $(a^x + a^{-x})^2$. | 59. $(9a^{-6} - 12a^{-3}b^{-2} + 4b^{-4})^{\frac{1}{2}}$. |
| 46. $(a^x - 7b^x)(a^x - 3b^x)$. | 60. $(x^2 + 2 + x^{-2})^{\frac{1}{2}}$. |
| 47. $(a^{-n} - b^{-n})^2$. | 61. $(a^{\frac{3}{7}} + 3a^{\frac{2}{7}} + 3a^{\frac{1}{7}} + 1)^{\frac{1}{3}}$. |
| 48. $(a^{-n} + 2a^n)^2$. | |

CHAPTER XV

RADICALS

250. A **radical** is the root of a quantity, indicated by a radical sign.

251. The radical is **rational**, if the root can be extracted exactly; **irrational**, if the root cannot be exactly obtained. Irrational quantities are frequently called *surds*.

$\sqrt{9}$, $(x + y)^{\frac{1}{2}}$ are radicals.

$4^{\frac{1}{2}} = 2$, $\sqrt{(a + b)^2}$ are rational.

$\sqrt{2}$, $\sqrt{4a + b}$ are irrational.

252. The **order** of a surd is indicated by the index of the root.

\sqrt{a} is of the second order, or quadratic.

$\sqrt[3]{2}$ is of the third order, or cubic.

$\sqrt[4]{c}$ is of the fourth order or biquadratic.

253. A **mixed surd** is the product of a rational factor and a surd factor; as $3\sqrt{a}$, $x\sqrt{3}$. The rational factor of a mixed surd is called the **coefficient** of the surd.

An **entire surd** is one whose coefficient is unity; as \sqrt{a} , $\sqrt{x^2 + y^2}$.

254. **Similar surds** are surds which contain the same irrational factor.

$3\sqrt{2}$ and $5a\sqrt{2}$ are similar.

$3\sqrt{2}$ and $3\sqrt{3}$ are dissimilar.

255. **Conventional restriction of the signs of roots.**

All even roots may be positive or negative,

e.g. $\sqrt{4} = +2$ or -2 .

Hence $5\sqrt{4} + 2\sqrt{4} = 5(\pm 2) + 2(\pm 2),$

which results in four values, viz. 14, 6, -14 , or -6 . To avoid this ambiguity, it is customary in elementary algebra to restrict the sign of a root to the prefixed sign.

$$\begin{aligned}\text{Thus} \quad 5\sqrt{4} + 2\sqrt{4} &= 7\sqrt{4} = 14. \\ 5\sqrt{2} - \sqrt{2} &= 3\sqrt{2}.\end{aligned}$$

If the object of an example, however, is merely an evolution, the complete answer is usually given; thus

$$\sqrt{x^2 - 4x + 4} = \pm (x - 2).$$

256. *Since radicals can be written as powers with fractional exponents, all examples relating to radicals may be solved by the methods employed for fractional exponents.*

Thus, to find the n th root of a product ab we have

$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}} \quad (\S 246).$$

I.e. to extract the root of a product, multiply the roots of the factors.

TRANSFORMATION OF RADICALS

257. Simplification of Surds. A radical is simplified when the expression under the radical sign is integral, and contains no factor whose power is equal to the index.

Ex. 1. Simplify $\sqrt{25a^4b}$.

$$\sqrt{25a^4b} = \sqrt{25a^4} \cdot \sqrt{b} = 5a^2\sqrt{b}.$$

Ex. 2. Simplify $\sqrt[3]{16}$.

$$\sqrt[3]{16} = \sqrt[3]{8} \cdot \sqrt[3]{2} = 2\sqrt[3]{2}.$$

Ex. 3. Simplify $\sqrt[4]{48a^5b^{10}}$.

$$\sqrt[4]{48a^5b^{10}} = \sqrt[4]{16a^4b^8} \cdot \sqrt[4]{3ab^2} = 2ab^2\sqrt[4]{3ab^2}.$$

258. When the quantity under the radical sign is a fraction, we multiply both numerator and denominator by such a quantity as will make the denominator a perfect power of the same degree as the surd.

Ex. 4. Simplify $\sqrt{\frac{1}{3}}$.

$$\sqrt{\frac{1}{3}} = \sqrt{\frac{1}{3} \cdot \frac{3}{3}} = \sqrt{\frac{3}{9}} = \frac{1}{3}\sqrt{3}.$$

Ex. 5. Simplify $\sqrt[3]{\frac{4x^2y^3}{9a^2b}}$.

$$\sqrt[3]{\frac{4x^2y^3}{9a^2b}} = \sqrt[3]{\frac{4x^2y^3}{9a^2b} \cdot \frac{3ab^2}{3ab^2}} = \sqrt[3]{\frac{12a^2b^2x^2y^3}{27a^3b^3}} = \frac{y}{3ab} \sqrt[3]{12a^2b^2x^2}.$$

EXERCISE 97

Simplify:

- | | | |
|--|---------------------------------|--------------------------------------|
| 1. $\sqrt{28}$. | 12. $\sqrt{243a^3b^2}$. | 23. $\sqrt[3]{250}$. |
| 2. $\sqrt{45}$. | 13. $3\sqrt{8}$. | 24. $\sqrt[3]{648}$. |
| 3. $\sqrt{18}$. | 14. $5\sqrt{80x^2}$. | 25. $2\sqrt[3]{-48a^2}$. |
| 4. $\sqrt{24}$. | 15. $8\sqrt{75a^2}$. | 26. $7\sqrt{108a^4b^4}$. |
| 5. $\sqrt{27}$. | 16. $6\sqrt{150a^2b^2}$. | 27. $5\sqrt{320a^5}$. |
| 6. $\sqrt{96}$. | 17. $3\sqrt{12a^3}$. | 28. $3\sqrt{16a^5b^5}$. |
| 7. $\sqrt{243}$. | 18. $7\sqrt{48ax^2}$. | 29. $\sqrt[5]{a^{10}b^{11}c^{12}}$. |
| 8. $\sqrt{320}$. | 19. $\frac{5}{2}\sqrt{24a^3}$. | 30. $\sqrt[n]{a^n b^{n+1}}$. |
| 9. $\sqrt{405a^2}$. | 20. $\frac{4}{3}\sqrt{27b^5}$. | 31. $\sqrt[n]{a^2nb^{3n+2}}$. |
| 10. $\sqrt{363x^2}$. | 21. $\sqrt[3]{80}$. | 32. $\sqrt{3(a+b)^2}$. |
| 11. $\sqrt{24a^3}$. | 22. $\sqrt[3]{-81}$. | |
| 33. $\sqrt{2(a^2+2ab+b^2)}$. | 34. $\sqrt{2x^2-12xy+18y^2}$. | |
| 35. $\sqrt{x^3-x^2}$. | 39. $\sqrt{\frac{1}{3}}$. | 42. $\sqrt{\frac{7}{3}}$. |
| 36. $(9a+18)^{\frac{1}{2}}$. | 40. $\sqrt{\frac{5x}{9}}$. | 43. $\sqrt{\frac{9}{8}}$. |
| 37. $(64a^6b^6c^{10})^{\frac{1}{5}}$. | | 44. $\sqrt{\frac{5}{14}}$. |
| 38. $\sqrt{\frac{1}{2}}$. | 41. $\sqrt{\frac{3}{5}}$. | 45. $\sqrt{\frac{19}{27}}$. |

$$46. a\sqrt{\frac{x}{a}}$$

$$47. b\sqrt{\frac{x^2}{b}}$$

$$48. c\sqrt{\frac{x^3}{c}}$$

$$49. c\sqrt[3]{\frac{x^2}{c}}$$

$$50. \sqrt[3]{\frac{1}{2}}$$

$$51. \sqrt[3]{\frac{4}{9}}$$

$$52. \sqrt{\frac{3}{5} \frac{a^2}{5}}$$

$$53. \sqrt[4]{\frac{9}{8}}$$

$$54. \sqrt{\frac{2}{3} \frac{a^2}{b^2}}$$

$$55. \sqrt{\frac{3}{5} \frac{a^2}{x^2}}$$

$$56. \sqrt{\frac{5}{6} \frac{a^3}{x^5}}$$

$$57. \frac{3}{a^2} \sqrt{\frac{13}{18} \frac{a^7}{x}}$$

$$58. \sqrt[3]{\frac{a}{b}}$$

$$59. \frac{2}{b} \frac{a}{\sqrt{\frac{3}{2} \frac{a^4}{b^4}}}$$

$$60. \frac{3}{2} \frac{x}{y^2} \sqrt{\frac{32}{3} \frac{y^4}{x^3}}$$

$$61. \sqrt{\frac{5}{81} \frac{a^5 b^6}{c^4}}$$

$$62. \sqrt{\frac{8}{(a+b)^3} \frac{a}{b}}$$

$$63. (a-b) \sqrt{\frac{a+b}{a-b}}$$

$$64. \frac{a}{b} \sqrt[n]{\frac{b^{n+1}}{a^{n-1}}}$$

$$65. a \sqrt[m]{\frac{b^m}{a^{m-1}}}$$

259. An imaginary surd can be simplified in precisely the same manner as a real surd; thus

$$\sqrt{-9} = 3\sqrt{-1}, \quad \sqrt{-\frac{a^2}{b^2}} = \frac{a}{b} \sqrt{-1}, \quad \sqrt{-\frac{4}{3} \frac{m^2}{3}} = \frac{2}{3} \frac{m}{3} \sqrt{-3}.$$

Simplify:

$$66. \sqrt{-a^2}.$$

$$68. \sqrt{-196}.$$

$$70. \frac{3}{x^2} \sqrt{-\frac{3}{9} \frac{x^3}{9}}.$$

$$67. \sqrt{-16 \frac{m}{m}}.$$

$$69. \sqrt{-\frac{2}{b} \frac{a}{b}}.$$

$$71. \sqrt{-a^2 - 2ab - b^2}.$$

260. Reduction of a surd to an entire surd.

Ex. 1. Express $4a\sqrt{b}$ as an entire surd.

$$4a\sqrt{b} = \sqrt{16a^2b} = \sqrt{16a^2b}.$$

Ex. 2. Express $\frac{a}{2b} \sqrt[n]{\frac{2^{n-1}b^{n+1}}{a^{n+1}}}$ as an entire surd.

$$\frac{a}{2b} \sqrt[n]{\frac{2^{n-1}b^{n+1}}{a^{n+1}}} = \sqrt[n]{\frac{a^n \cdot 2^{n-1} \cdot b^{n+1}}{2^n b^n a^{n+1}}} = \sqrt[n]{\frac{b}{2a}}.$$

EXERCISE 98

Express as entire surds:

1. $2\sqrt{3}$.

9. $\frac{4}{5}\sqrt{15a^3}$.

14. $\frac{a+b}{a-b}\sqrt{\frac{a-b}{a+b}}$.

2. $3\sqrt[3]{2}$.

10. $\frac{a}{c}\sqrt[4]{\frac{6c}{5a^3}}$.

15. $\frac{m}{n}\sqrt[n]{\frac{n^{a+1}}{m^{a+1}}}$.

3. $3\sqrt[4]{2}$.

11. $\frac{2a}{3b}\sqrt[3]{\frac{27b}{a^4}}$.

16. $\frac{a}{b^n}\sqrt[3]{\frac{9b^{2n+1}}{a}}$.

4. $2\sqrt[5]{3}$.

5. $\frac{1}{2}\sqrt{6}$.

6. $\frac{3}{4}\sqrt[3]{4}$.

12. $\frac{bc}{d}\sqrt[4]{4d}$.

17. $\frac{2x}{y}\sqrt[n]{\frac{y^{2n}}{2^n x^{n+1}}}$.

7. $a^2\sqrt{b}$.

8. $\frac{3}{x}\sqrt{x^4}$.

13. $\frac{a}{b}\sqrt[6]{\frac{b^7}{a^7}}$.

18. $\frac{m+n}{m-n}\sqrt[3]{m^2-n^2}$.

261. Transformation of surds to surds of different order.

Ex. 1. Transform $\sqrt[4]{a^3b^2}$ into a surd of the 20th order.

$$\sqrt[4]{a^3b^2} = a^{\frac{3}{4}}b^{\frac{2}{4}} = a^{\frac{15}{20}}b^{\frac{10}{20}} = \sqrt[20]{a^{15}b^{10}}.$$

Ex. 2. Transform $\sqrt{2}$, $\sqrt[3]{3}$, and $\sqrt[4]{5}$ into surds of the same lowest order.

$$\sqrt{2} = 2^{\frac{1}{2}} = 2^{\frac{6}{12}} = \sqrt[12]{64}.$$

$$\sqrt[3]{3} = 3^{\frac{1}{3}} = 3^{\frac{4}{12}} = \sqrt[12]{81}.$$

$$\sqrt[4]{5} = 5^{\frac{1}{4}} = 5^{\frac{3}{12}} = \sqrt[12]{125}.$$

Ex. 3. Reduce the order of the surd $\sqrt[16]{a^{12}}$.

$$\sqrt[16]{a^{12}} = a^{\frac{12}{16}} = a^{\frac{3}{4}} = \sqrt[4]{a^3}.$$

Exponent and index bear the same relation as numerator and denominator of a fraction; and hence both may be multiplied by the same number, or both divided by the same number, without changing the value of the radical.

EXERCISE 99

Reduce to surds of the 6th order :

- | | | | |
|---------------------|----------------------------|---------------------|---------------------------|
| 1. \sqrt{x} . | 3. $\sqrt{\frac{xy}{z}}$. | 5. $\sqrt[6]{10}$. | 7. $\sqrt[12]{a^{-10}}$. |
| 2. $\sqrt[3]{ab}$. | 4. ab . | 6. 4. | 8. $\sqrt[18]{x^{12}}$. |

Reduce to surds of the 12th order :

- | | | | |
|--------------------------|-----------------------------|---------------------|--------------------------|
| 9. $\sqrt{a^2b^2}$. | 11. $\sqrt[3]{a^4b^4c^4}$. | 13. 2. | 15. $\sqrt[6]{a^{-2}}$. |
| 10. $\sqrt[4]{a^3b^2}$. | 12. abc^2 . | 14. $\sqrt[6]{4}$. | 16. $\sqrt[3]{m^2}$. |

Express as surds of lowest order with integral exponents and indices :

- | | | | |
|------------------------|-----------------------------|----------------------------|---|
| 17. $\sqrt[12]{a^9}$. | 21. $\sqrt[6]{8}$. | 25. $\sqrt[6]{27a^3b^6}$. | 29. $\sqrt[6]{\frac{a^3b^3}{c^6}}$. |
| 18. $\sqrt[4]{4}$. | 22. $\sqrt[6]{a^2b^2}$. | 26. $\sqrt[4]{81a^2b^4}$. | |
| 19. $\sqrt[6]{27}$. | 23. $\sqrt[4]{x^2y^2}$. | 27. $\sqrt[40]{a^{20}}$. | 30. $\sqrt[4]{\frac{25a^2}{(a+b)^2}}$. |
| 20. $\sqrt[4]{49}$. | 24. $\sqrt[10]{32x^5y^5}$. | 28. $\sqrt[14]{a^7b^7}$. | |

Express as surds of the same lowest order :

- | | |
|----------------------------------|--|
| 31. $\sqrt{2}, \sqrt[3]{3}$. | 37. $2^{\frac{1}{3}}, 3^{\frac{1}{4}}$. |
| 32. $\sqrt[4]{a}, \sqrt[3]{b}$. | 38. $\sqrt[5]{a^2}, \sqrt[6]{a^5}, \sqrt[3]{a^2}$. |
| 33. $\sqrt[4]{4}, \sqrt[6]{5}$. | 39. $\sqrt{3}, \sqrt[3]{3}, \sqrt[4]{4}$. |
| 34. $\sqrt[3]{7}, \sqrt[4]{9}$. | 40. $\sqrt[2]{8}, \sqrt[3]{8}, \sqrt[3]{7}$. |
| 35. $\sqrt[4]{2}, \sqrt[3]{3}$. | 41. $\sqrt[3]{2}, \sqrt[9]{8}, \sqrt[6]{4}$. |
| 36. $\sqrt{2}, \sqrt[5]{5}$. | 42. $\sqrt[5]{a^{10}b^{10}}, \sqrt[3]{a^2b^2}, \sqrt[6]{a^5b^7}$. |

Arrange in order of magnitude :

- | | |
|--|---|
| 43. $\sqrt{2}, \sqrt[3]{3}$. | 47. $\sqrt[4]{10}, \sqrt[10]{901}$. |
| 44. $\sqrt[3]{4}, \sqrt[4]{6}$. | 48. $\sqrt{\frac{2}{3}}, \sqrt[3]{\frac{3}{4}}, \sqrt[6]{\frac{7}{27}}$. |
| 45. $\sqrt[5]{7}, \sqrt[2]{2}$. | 49. $5\sqrt{2}, 4\sqrt[3]{4}$. |
| 46. $\sqrt[5]{5}, \sqrt[3]{11}, \sqrt[6]{124}$. | 50. $4\sqrt[9]{8}, 2\sqrt[6]{9}, 3\sqrt{15}$. |

ADDITION AND SUBTRACTION OF RADICALS

262. To add or subtract surds, reduce them to their simplest form. If the resulting surds are similar, add them like similar terms (i.e. add their coefficients); if dissimilar, connect them by their proper signs.

Ex. 1. Simplify $\sqrt{\frac{1}{2}} + 3\sqrt{18} - 2\sqrt{50}$.

$$\sqrt{\frac{1}{2}} + 3\sqrt{18} - 2\sqrt{50} = \frac{1}{2}\sqrt{2} + 9\sqrt{2} - 10\sqrt{2} = -\frac{1}{2}\sqrt{2}.$$

Ex. 2. Simplify $\sqrt[3]{a^3x} - 3\sqrt[3]{\frac{b^6}{x^2}} + \sqrt[3]{\frac{27x}{y^3}}$.

$$\sqrt[3]{a^3x} - 3\sqrt[3]{\frac{b^6}{x^2}} + \sqrt[3]{\frac{27x}{y^3}} = a\sqrt[3]{x} - \frac{3b^2}{x}\sqrt[3]{x} + \frac{3}{y}\sqrt[3]{x} = \left(a - \frac{3b^2}{x} + \frac{3}{y}\right)\sqrt[3]{x}.$$

Ex. 3. Simplify $\sqrt{\frac{9}{2}} - \sqrt[3]{\frac{1}{4}} + \sqrt{72} - 4\sqrt{\frac{1}{3}} + \sqrt[3]{16}$.

$$\begin{aligned}\sqrt{\frac{9}{2}} - \sqrt[3]{\frac{1}{4}} + \sqrt{72} - 4\sqrt{\frac{1}{3}} + \sqrt[3]{16} &= \frac{3}{2}\sqrt{2} - \frac{1}{2}\sqrt[3]{2} + 6\sqrt{2} - \frac{4}{3}\sqrt{3} + 2\sqrt[3]{2} \\ &= \frac{15}{2}\sqrt{2} + \frac{3}{2}\sqrt[3]{2} - \frac{4}{3}\sqrt{3}.\end{aligned}$$

EXERCISE 100

Simplify the following expressions:

1. $\sqrt{8} - \sqrt{32} - \sqrt{72}$.
2. $\sqrt{18} - \sqrt{32} - \sqrt{50}$.
3. $\sqrt{128} - \sqrt{32} - \sqrt{18}$.
4. $\sqrt{45} - \sqrt{245} + \sqrt{180}$.
5. $\sqrt{80} - \sqrt{20} - \sqrt{5}$.
6. $\sqrt{63} + \sqrt{700} - \sqrt{175}$.
7. $3\sqrt{20} + 5\sqrt{45} - 4\sqrt{80}$.
8. $3\sqrt{8} + 2\sqrt{32} + 4\sqrt{72}$.
9. $2\sqrt{150} - 4\sqrt{54} + 6\sqrt{24}$.
10. $2\sqrt{32} + \frac{1}{5}\sqrt{50} + \frac{1}{6}\sqrt{72}$.
11. $\frac{3}{2}\sqrt{44} + 4\sqrt{99} + 5\sqrt{176}$.
12. $2\sqrt{125} - \frac{3}{2}\sqrt{45} + 4\sqrt{20}$.
13. $\sqrt[3]{16} - \sqrt[3]{54} + \sqrt[3]{2}$.
14. $2\sqrt[3]{16} + 3\sqrt[3]{250} - 4\sqrt[3]{128}$.
15. $4\sqrt[3]{54} + 5\sqrt[3]{250} + 2\sqrt[3]{16}$.
16. $3\sqrt[3]{250} - 2\sqrt[3]{686} + 2\sqrt[3]{1458}$.
17. $4\sqrt{a^2x} + 3\sqrt{b^2x} - 2a\sqrt{x}$.
18. $3\sqrt{a^3b^3} - 3a\sqrt{ab^3} - 3ab\sqrt{ab}$.

$$19. \sqrt{3} + \sqrt{\frac{1}{3}} + \sqrt{27}.$$

$$21. \sqrt{6a} + \sqrt{\frac{3a}{2}} - \sqrt{24a}.$$

$$20. \sqrt{5} - \sqrt{20} + \sqrt{\frac{1}{5}}.$$

$$22. \sqrt{120} + \sqrt{\frac{5}{6}} + \frac{1}{2}\sqrt{\frac{6}{5}}.$$

$$23. \sqrt{32} + 3\sqrt{\frac{1}{8}} + 4\sqrt{72}.$$

$$24. \sqrt{\frac{ab}{2}} - \sqrt{\frac{ab}{8}} + \sqrt{\frac{ab}{32}} + \sqrt{\frac{ab}{128}}.$$

$$25. \frac{1}{2}\sqrt{\frac{5}{8}} + \frac{1}{5}\sqrt{40} + 3\sqrt{\frac{5}{2}} - \sqrt{\frac{125}{8}}.$$

$$26. 3\sqrt{\frac{3}{7x}} - 4\sqrt{\frac{7}{3x}} + \frac{20}{21}\sqrt{84x}.$$

$$27. \sqrt{1\frac{1}{3}} + 5\sqrt{5\frac{1}{3}} - 3\sqrt{8\frac{1}{3}}.$$

$$28. 3\sqrt{3\frac{1}{5}} - 5\sqrt{9\frac{4}{5}} - 7\sqrt{125}.$$

$$29. \frac{2}{5}\sqrt[3]{1\frac{7}{9}} - \frac{1}{2}\sqrt[3]{14\frac{2}{9}} + \frac{2}{3}\sqrt[3]{6}.$$

$$30. \frac{5a}{6}\sqrt[3]{\frac{x}{5}} + \frac{a}{15b}\sqrt[3]{200b^3x} - 4\sqrt[3]{\frac{25a^3x}{512}}.$$

$$31. \sqrt{2x} + \sqrt[3]{16x} + \sqrt{50x} + \sqrt[3]{2x}.$$

$$32. \sqrt[3]{128ab} + \sqrt{72ab} - \sqrt{50ab} + \sqrt[3]{54ab}.$$

$$33. \sqrt[4]{81a^5} + 2\sqrt[4]{16a^5} - \sqrt[4]{256a^5}.$$

$$34. \sqrt[4]{a^2} - \sqrt[2]{a} + \sqrt{(a+b)^2a}.$$

$$35. \sqrt{4+4x^2} + \sqrt{9+9x^2} - 5\sqrt{1+x^2}.$$

$$36. \sqrt{a^2x} + \sqrt{b^2x} + \sqrt{c^2x}.$$

$$37. \sqrt[6]{a^3b^3} + \sqrt[4]{a^2b^2} - \sqrt{4ab}.$$

$$38. \sqrt{\frac{a^4c}{b^3}} + \sqrt{\frac{a^2c^3}{bx^2}} + \sqrt{\frac{a^2cx^2}{by^2}}.$$

$$39. \sqrt[3]{a^2x^4} + \sqrt[3]{27a^5x} - \sqrt[3]{125a^2x}.$$

$$40. \sqrt[10]{x^5} + \sqrt[8]{\frac{1}{x^4}}.$$

$$41. \sqrt{-9} + \sqrt{-16} - \sqrt{-25}.$$

$$42. \sqrt{-12a^2} + \sqrt{-75a^2} - \sqrt{-48a^2}.$$

MULTIPLICATION OF RADICALS

263. *Surds of the same order are multiplied by multiplying the product of the coefficients by the product of the irrational factors, for $a\sqrt[n]{x} \cdot b\sqrt[n]{y} = ab\sqrt[n]{xy}$.*

Dissimilar surds are reduced to surds of the same order, and then multiplied.

Ex. 1. Multiply $3\sqrt[3]{25y^2}$ by $5\sqrt[3]{50y^2}$.

$$3\sqrt[3]{25y^2} \cdot 5\sqrt[3]{50y^2} = 15\sqrt[3]{5^2 \cdot 2 \cdot 5^2 \cdot y^4} = 75y\sqrt[3]{10y}.$$

Ex. 2. Multiply $\sqrt{2}$ by $3\sqrt[3]{4}$.

$$\sqrt{2} \cdot 3\sqrt[3]{4} = \sqrt[6]{2^3} \cdot 3\sqrt[6]{4^2} = \sqrt[6]{2^3} \cdot 3\sqrt[6]{2^4} = 3\sqrt[6]{2^7} = 6\sqrt[6]{2}.$$

Ex. 3. Multiply $5\sqrt{7} - 2\sqrt{5}$ by $3\sqrt{7} + 10\sqrt{5}$.

$$\begin{array}{r} 5\sqrt{7} - 2\sqrt{5} \\ 3\sqrt{7} + 10\sqrt{5} \\ \hline 105 - 6\sqrt{35} \\ \quad + 50\sqrt{35} - 100 \\ \hline 105 + 44\sqrt{35} - 100 = 5 + 44\sqrt{35} \end{array}$$

EXERCISE 101

Simplify:

- | | |
|--|--|
| 1. $\sqrt{7} \cdot \sqrt{42}$. | 11. $7\sqrt{5x} \cdot \sqrt{x}$. |
| 2. $\sqrt{10} \cdot \sqrt{5}$. | 12. $3\sqrt{c^7} \cdot \sqrt{7c}$. |
| 3. $2\sqrt{3} \cdot \sqrt{12}$. | 13. $4\sqrt[3]{2d^2} \cdot \sqrt[3]{4d}$. |
| 4. $2\sqrt{28} \cdot \sqrt{7}$. | 14. $3a^4\sqrt{a^3} \cdot \sqrt[4]{a^5}$. |
| 5. $\sqrt[3]{4} \cdot \sqrt[3]{2}$. | 15. $6x\sqrt[6]{x^7y} \cdot \sqrt[6]{x^2y}$. |
| 6. $\sqrt[3]{25} \cdot \sqrt[3]{5}$. | 16. $\sqrt{xy} \cdot \sqrt{\frac{x}{y}}$. |
| 7. $3\sqrt[3]{49} \cdot \sqrt[3]{7}$. | 17. $\sqrt{a} \cdot \sqrt{\frac{5x}{4}}$. |
| 8. $3\sqrt[3]{2} \cdot \sqrt[3]{4}$. | 18. $(\sqrt{3} + \sqrt{4} + \sqrt{5})\sqrt{5}$. |
| 9. $a\sqrt{a} \cdot x\sqrt{x}$. | |
| 10. $5\sqrt{ax} \cdot 3\sqrt{5x}$. | |

19. $(2\sqrt{3} + 3\sqrt{4} + 4\sqrt{5})\sqrt{3}$.
20. $(3\sqrt{5} + 2\sqrt{3} + \sqrt{6})\sqrt{6}$.
21. $(a + \sqrt{b})(a - \sqrt{b})$.
22. $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.
23. $(\sqrt{3a} + \sqrt{2b})(\sqrt{3a} - \sqrt{2b})$.
24. $(3\sqrt{5} + 2\sqrt{11})(3\sqrt{5} - 2\sqrt{11})$.
25. $(\sqrt{x+y} + \sqrt{y})(\sqrt{x+y} - \sqrt{y})$.
26. $(\sqrt{x} + \sqrt{x-y})(\sqrt{x} - \sqrt{x-y})$.
27. $(\sqrt{9x+5} + 3\sqrt{x})(\sqrt{9x+5} - 3\sqrt{x})$.
28. $\left(\sqrt{\frac{a+1}{2}} + \sqrt{\frac{a-1}{2}}\right)\left(\sqrt{\frac{a+1}{2}} - \sqrt{\frac{a-1}{2}}\right)$.
29. $(\sqrt{a} + \sqrt{b})^2$.
30. $(\sqrt{3} + \sqrt{2})^2$.
31. $(1 + \sqrt{2})^2$.
32. $(\sqrt{6} - \sqrt{5})^2$.
33. $(\sqrt{x+y} + \sqrt{x-y})^2$.
34. $(a + \sqrt{1-a^2})^2$.
35. $(-1 + \sqrt{3})^2$.
36. $(3\sqrt{2} - 2\sqrt{5})^2$.
37. $\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2$.
38. $\left(\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}}\right)^2$.
39. $(\sqrt{a} + \sqrt{b})^3$.
40. $(\sqrt[3]{3} - \sqrt[3]{2})^3$.
41. $(\sqrt{a} + \sqrt{b})(a + b - \sqrt{ab})$.
42. $(\sqrt{a} + \sqrt{b} + \sqrt{c})^2$.
43. $(\sqrt{2} + \sqrt{5} - \sqrt{10})^2$.
44. $\sqrt{a^b} \cdot \sqrt{a^{2-b}} \cdot \sqrt{a}$.
45. $\sqrt[3]{a^{c+1}} \cdot \sqrt[3]{a^{1-c}} \cdot \sqrt[3]{a}$.
46. $\sqrt[3]{\frac{1}{2}} \cdot \sqrt[3]{\frac{2}{3}} \cdot \sqrt[3]{\frac{3}{4}}$.
47. $\sqrt[n]{a^{n-2}b^3} \cdot \sqrt[n]{b^{n-3}a^2}$.
48. $\sqrt[6]{a^3+b^3} \cdot \sqrt[6]{a^3-b^3}$.
49. $(\sqrt{7} - \sqrt{3})(\sqrt{3} - \sqrt{2})$.
50. $(3\sqrt{2} - 2\sqrt{3})(7\sqrt{2} + 5\sqrt{3})$.
51. $(5\sqrt{3} + \sqrt{6})(5\sqrt{2} - 2)$.
52. $(2\sqrt{6} + 5\sqrt{3} - 7\sqrt{2})(3\sqrt{3} - \sqrt{2})$.
53. $(3 + \sqrt{6} + \sqrt{15})(2 + \sqrt{6} - \sqrt{10})$.
54. $(\sqrt[3]{3} + \sqrt[3]{2})(2\sqrt[3]{9} - 3\sqrt[3]{4})$.
55. $\sqrt[5]{a} \cdot \sqrt[5]{a}$.
56. $\sqrt[3]{c} \cdot \sqrt[6]{c}$.
57. $\sqrt[3]{a} \cdot \sqrt[4]{\frac{b}{a}}$.
58. $\sqrt{\frac{m}{n}} \cdot \sqrt[3]{\frac{n}{m}}$.

59. $\sqrt[3]{\frac{2}{3}} \cdot \sqrt{6}.$

64. $\sqrt[3]{c} \cdot \sqrt[4]{c} \cdot \sqrt[5]{10c}.$

60. $\sqrt[3]{\frac{3}{8}} \cdot \sqrt{\frac{4}{3}}.$

65. $(\sqrt{6} + \sqrt[3]{4})(\sqrt{6} - \sqrt[3]{9}).$

61. $\sqrt[m]{a} \cdot \sqrt[n]{b}.$

66. $(\sqrt{5} + \sqrt[3]{25})^2.$

62. $\sqrt[n]{x} \cdot \sqrt[2n]{y}.$

67. $(\sqrt{2} + \sqrt[3]{4})^2.$

63. $\sqrt[4]{2x} \cdot \sqrt[5]{\frac{3y}{2x}}.$

DIVISION OF RADICALS

264. *Monomial surds of the same order may be divided by multiplying the quotient of the coefficients by the quotient of the surd factors.* E.g. $a\sqrt{b} \div x\sqrt{y} = \frac{a}{x} \sqrt{\frac{b}{y}}.$

Since surds of different orders can be reduced to surds of the same order, all monomial surds may be divided by this method.

Ex. 1. $4\sqrt{48} \div 3\sqrt{6} = \frac{4}{3}\sqrt{8} = \frac{8}{3}\sqrt{2}.$

Ex. 2. $(\sqrt{50} + 3\sqrt{12}) \div \sqrt{2} = \sqrt{25} + 3\sqrt{6} = 5 + 3\sqrt{6}.$

265. If, however, the quotient of the surds is a fraction, it is more convenient to multiply dividend and divisor by a factor which makes the divisor rational.

This method, called **rationalizing the divisor**, is illustrated by the following examples:

Ex. 1. Divide $\sqrt{11}$ by $\sqrt{7}.$

In order to make the divisor ($\sqrt{7}$) rational, we have to multiply by $\sqrt{7}.$

$$\frac{\sqrt{11}}{\sqrt{7}} = \frac{\sqrt{11}}{\sqrt{7}} \cdot \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{77}}{7} = \frac{1}{7}\sqrt{77}.$$

Ex. 2. Divide $4\sqrt[3]{3a}$ by $3\sqrt[3]{2b^2}.$

The rationalizing factor is evidently $\sqrt[3]{4b}$; hence,

$$\frac{4\sqrt[3]{3a}}{3\sqrt[3]{2b^2}} = \frac{4\sqrt[3]{3a}}{3\sqrt[3]{2b^2}} \cdot \frac{\sqrt[3]{4b}}{\sqrt[3]{4b}} = \frac{4\sqrt[3]{12ab}}{3 \cdot 2b} = \frac{2\sqrt[3]{12ab}}{3b}.$$

Ex. 3. Simplify $\frac{1}{4\sqrt[5]{a^3}\sqrt[7]{b^2}}$.

$$\frac{1}{4\sqrt[5]{a^3}\sqrt[7]{b^2}} = \frac{1}{4\sqrt[5]{a^3}\sqrt[7]{b^2}} \cdot \frac{\sqrt[5]{a^2}\sqrt[7]{b^5}}{\sqrt[5]{a^2}\sqrt[7]{b^5}} = \frac{\sqrt[5]{a^2}\sqrt[7]{b^5}}{4ab}.$$

Ex. 4. Divide $12\sqrt{3} + 4\sqrt{5}$ by $\sqrt{8}$.

Since $\sqrt{8} = 2\sqrt{2}$, the rationalizing factor is $\sqrt{2}$,

$$\frac{12\sqrt{3} + 4\sqrt{5}}{\sqrt{8}} = \frac{12\sqrt{3} + 4\sqrt{5}}{\sqrt{8}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{12\sqrt{6} + 4\sqrt{10}}{4} = 3\sqrt{6} + \sqrt{10}.$$

266. To show that expressions with rational denominators are simpler than those with irrational denominators, arithmetical problems afford the best illustrations. To find, *e.g.* $\frac{1}{\sqrt{3}}$ by the usual arithmetical method, we have

$$\frac{1}{\sqrt{3}} = \frac{1}{1.73205}.$$

But if we simplify $\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = \frac{1.73205}{3}$.

Either quotient equals .57735. Evidently, however, the division by 3 is much easier to perform than the division by 1.73205. Hence in arithmetical work it is always best to rationalize the denominators before dividing.

EXERCISE 102

Simplify:

1. $\sqrt{12} \div \sqrt{6}.$

4. $\sqrt{6x} \div \sqrt{2x}.$

2. $\sqrt{18} \div \sqrt{2}.$

5. $\sqrt[3]{81} \div \sqrt[3]{3}.$

3. $\sqrt{2x} \div \sqrt{x}.$

6. $(4\sqrt{6} + 4\sqrt{2}) \div \sqrt{2}.$

7. $(3\sqrt[3]{12} + 5\sqrt[3]{16} - 2\sqrt[3]{28}) \div \sqrt[3]{4}.$

8. $\sqrt{\frac{2}{3}} \div \sqrt{\frac{3}{2}}.$

11. $\sqrt{ab} \div \sqrt{\frac{a}{b}}.$

9. $\sqrt{\frac{a}{b}} \div \sqrt{\frac{b}{a}}.$

12. $\sqrt[3]{2x^4y} \div \sqrt[3]{\frac{x}{4y^2}}.$

10. $\sqrt{x^2 - y^2} \div \sqrt{x - y}.$

13. $\frac{a}{\sqrt{a}}.$

18. $\frac{9}{2\sqrt{3}}.$

23. $\frac{5(1+a^2)}{3\sqrt[3]{1+a^2}}.$

14. $\frac{a}{\sqrt[3]{a}}.$

19. $\frac{48}{5\sqrt{32}}.$

24. $\frac{a}{\sqrt[7]{a^5}}.$

15. $\frac{3}{\sqrt{3}}.$

20. $\frac{a}{\sqrt[n]{a}}.$

25. $\frac{a}{\sqrt[8]{a^3}}.$

16. $\frac{4}{\sqrt{2}}.$

21. $\frac{a}{\sqrt[n]{a^{n-1}}}.$

26. $\frac{a^2-b^2}{\sqrt{a-b}}.$

17. $\frac{1}{\sqrt{7}}.$

22. $\frac{x+y}{\sqrt{x+y}}.$

Given $\sqrt{2}=1.41421$, $\sqrt{3}=1.73205$, $\sqrt{5}=2.23607$, and $\sqrt{6}=2.44949$, find to four decimal places the numerical values of:

27. $\frac{1}{\sqrt{2}}.$

29. $\frac{3}{\sqrt{3}}.$

31. $\frac{6}{\sqrt{8}}.$

28. $\frac{1}{\sqrt{3}}.$

30. $\frac{5}{\sqrt{5}}.$

32. $\frac{4}{\sqrt{50}}.$

33. $\frac{2}{\sqrt{125}}.$

34. $\frac{12}{\sqrt{6}}.$

35. $\frac{2}{\sqrt{20}}.$

36. $\frac{12}{\sqrt{45}}.$

267. Two binomial quadratic surds are said to be **conjugate**, if they differ only in the sign which connects their terms.

$\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are conjugate surds.

268. *The product of two conjugate binomial surds is rational.*

$$(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b.$$

269. To rationalize the denominator of a fraction whose denominator is a binomial quadratic surd, multiply numerator and denominator by the conjugate surd of the denominator.

Ex. 1. Simplify $\frac{2\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

$$\frac{2\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} = \frac{2\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} \cdot \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}} = \frac{4+\sqrt{6}}{3-2} = 4+\sqrt{6}.$$

Ex. 2. Simplify $\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}}$.

$$\frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} = \frac{x-\sqrt{x^2-1}}{x+\sqrt{x^2-1}} \cdot \frac{x-\sqrt{x^2-1}}{x-\sqrt{x^2-1}} = \frac{x^2-2x\sqrt{x^2-1}+x^2-1}{x^2-(x^2-1)} = \frac{2x^2-1-2x\sqrt{x^2-1}}{1}.$$

Ex. 3. Find the numerical value of:

$$\frac{\sqrt{2}+2}{2\sqrt{2}-1}.$$

$$\frac{\sqrt{2}+2}{2\sqrt{2}-1} = \frac{\sqrt{2}+2}{2\sqrt{2}-1} \cdot \frac{2\sqrt{2}+1}{2\sqrt{2}+1} = \frac{6+5\sqrt{2}}{7} = \frac{13.07105}{7} = 1.8673.$$

270. If the denominator is a *trinomial quadratic surd*, two multiplications are necessary to rationalize the denominator.

J. Ex. 4. Rationalize the denominator of:

$$\frac{1+\sqrt{2}+\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}.$$

$$\frac{1+\sqrt{2}+\sqrt{3}}{1+\sqrt{2}-\sqrt{3}} = \frac{1+\sqrt{2}+\sqrt{3}}{(1+\sqrt{2})-\sqrt{3}} \cdot \frac{1+\sqrt{2}+\sqrt{3}}{(1+\sqrt{2})+\sqrt{3}} = \frac{6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}}{(1+2\sqrt{2}+2)-3}$$

$$= \frac{3+\sqrt{2}+\sqrt{3}+\sqrt{6}}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{3\sqrt{2}+2+\sqrt{6}+2\sqrt{3}}{2}.$$

EXERCISE 103

Rationalize the denominator of:

1. $\frac{1}{2+\sqrt{2}}$

3. $\frac{\sqrt{6}}{4-2\sqrt{3}}$

5. $\frac{5\sqrt{6}}{7-3\sqrt{2}}$

2. $\frac{2}{2-\sqrt{3}}$

4. $\frac{9\sqrt{10}}{4+2\sqrt{5}}$

6. $\frac{11\sqrt{15}}{10-3\sqrt{5}}$

7. $\frac{2 - \sqrt{2}}{2 + \sqrt{2}}$

12. $\frac{3\sqrt{7} - 5\sqrt{3}}{3\sqrt{7} + 5\sqrt{3}}$

17. $\frac{1}{\sqrt{x} + \sqrt{y}}$

8. $\frac{3 + \sqrt{3}}{3 - \sqrt{3}}$

13. $\frac{5\sqrt{3} - 3\sqrt{5}}{\sqrt{5} - \sqrt{3}}$

18. $\frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$

9. $\frac{24}{\sqrt{8} - 2}$

14. $\frac{7\sqrt{5} + 5\sqrt{7}}{\sqrt{7} + \sqrt{5}}$

19. $\frac{5 + \sqrt{x}}{5 - \sqrt{x}}$

10. $\frac{\sqrt{5} - \sqrt{3}}{\sqrt{3} + \sqrt{2}}$

15. $\frac{\sqrt{6} - \sqrt{3}}{\sqrt{3} - \sqrt{2}}$

20. $\frac{a + b\sqrt{x}}{c + d\sqrt{x}}$

11. $\frac{7\sqrt{2} - 5\sqrt{3}}{4\sqrt{3} - 3\sqrt{2}}$

16. $\frac{a}{a + \sqrt{a}}$

21. $\frac{ax - bx}{\sqrt{a} + \sqrt{b}}$

22. $\frac{2}{a - \sqrt{a^2 - 4}}$

23. $\frac{2ax}{\sqrt{2ax} + x\sqrt{a}}$

24. $\frac{\sqrt{a+b} + \sqrt{a-b}}{\sqrt{a+b} - \sqrt{a-b}}$

25. $\frac{a^2}{a + \sqrt{a^2 - b^2}}$

26. $\frac{1}{3^{\frac{1}{2}} - 1}$

27. $\frac{2\sqrt{2}}{1 + \sqrt{2} - \sqrt{3}}$

28. $\frac{1}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$

29. $\frac{1 + \sqrt{2}}{1 + 2\sqrt{2} + 3\sqrt{3}}$

30. $\frac{1}{2\sqrt{1} + 1 - \sqrt{5}}$

31. $\frac{1}{\sqrt{\sqrt{2} - 1}}$

Given $\sqrt{2} = 1.41421$, $\sqrt{3} = 1.73205$, and $\sqrt{5} = 2.23607$, find to four places of decimals:

32. $\frac{1}{1 + \sqrt{2}}$

34. $\frac{4}{3 - \sqrt{5}}$

36. $\frac{\sqrt{3}}{2 - \sqrt{3}}$

33. $\frac{2}{2 - \sqrt{3}}$

35. $\frac{\sqrt{6} - \sqrt{3}}{\sqrt{6} + \sqrt{3}}$

37. $\frac{1}{\sqrt{12} - \sqrt{8}}$

INVOLUTION AND EVOLUTION OF RADICALS

271. Powers and roots of radicals can be found by the use of fractional exponents.

Ex. 1. Find $\sqrt[3]{\sqrt[4]{a}}$.

$$\sqrt[3]{\sqrt[4]{a}} = (a^{\frac{1}{4}})^{\frac{1}{3}} = a^{\frac{1}{12}} = \sqrt[12]{a}.$$

Ex. 2. Find the square of $\sqrt[3]{2x^2}$.

$$(\sqrt[3]{2x^2})^2 = (2x^2)^{\frac{2}{3}} = \sqrt[3]{4x^4} = x\sqrt[3]{4x}.$$

272. A case which frequently occurs is expressed by the formula $\sqrt[n]{a^m} = (\sqrt[n]{a})^m$. This equation is correct since each member equals $a^{\frac{m}{n}}$. It should be remembered, however, that the signs of the two members are equal only if they are restricted in the manner of Article 255. If the radicals are taken with all possible signs, $\sqrt[n]{a^m}$ does not necessarily equal $(\sqrt[n]{a})^m$, as illustrated by the following example:

$(\sqrt{a})^2 = a$, by definition of square root,

$$\sqrt{a^2} = \pm a.$$

Ex. 3. Find $(2\sqrt[3]{a^2x^5})^4$.

$$(2\sqrt[3]{a^2x^5})^4 = 2^4\sqrt[3]{a^8x^{20}} = 16a^2x^6\sqrt[3]{a^2x^2}.$$

Ex. 4. Find $\sqrt[2]{144^3}$.

$$\sqrt[2]{144^3} = (\sqrt[2]{144})^3 = (\pm 12)^3 = \pm 1728.$$

EXERCISE 104

Simplify:

1. $(2\sqrt{x})^2$.

5. $(4a\sqrt[3]{x})^6$.

9. $\sqrt[2]{4\sqrt{x}}$.

2. $(2\sqrt{x})^3$.

6. $\sqrt[3]{8^5}$.

10. $\sqrt{\sqrt{x}}$.

3. $(2\sqrt{x})^4$.

7. $\sqrt[3]{125^4}$.

11. $\sqrt[3]{\sqrt[2]{x}}$.

4. $(2a\sqrt[3]{x^2})^4$.

8. $\sqrt[4]{16^6}$.

12. $\sqrt[3]{\sqrt[2]{x^3y^3z^3}}$.

SQUARE ROOTS OF QUADRATIC SURDS

273. *A quadratic surd cannot be equal to the sum of a rational expression and a quadratic surd.*

For, if possible, let $\sqrt{a} = b + \sqrt{c}$.

Squaring both members, $a = b^2 + 2b\sqrt{c} + c$.

Solving for \sqrt{c} , $\sqrt{c} = \frac{a - b^2 - c}{2b}$.

That is, a quadratic surd is equal to a rational expression, which is impossible.

274. If $a + \sqrt{b} = x + \sqrt{y}$, then $a = x$ and $b = y$.

Suppose a were not equal to x .

Transposing a , $\sqrt{b} = x - a + \sqrt{y}$,

which is impossible. (§ 273.)

Therefore $a = x$, and hence $b = y$.

275. To find the square root of a binomial surd.

To find the square root of $17 + \sqrt{240}$.

Assume $\sqrt{17 + \sqrt{240}} = \sqrt{x} + \sqrt{y}$.

Squaring both members, $17 + \sqrt{240} = x + 2\sqrt{xy} + y$.

Then, by § 274, $x + y = 17$. (1)

$2\sqrt{xy} = \sqrt{240}$. (2)

The solution of (1) and (2) gives the values of x and y . The most convenient method for solving consists in subtracting the square of (2) from the square of (1), thus obtaining $(x - y)^2$.

Squaring (1), $x^2 + 2xy + y^2 = 289$. (3)

Squaring (2), $4xy = 240$. (4)

(3)-(4), $x^2 - 2xy + y^2 = 49$. (5)

Extracting square roots, $x - y = 7$. (6)

But $x + y = 17.$ (1)

Therefore $x = 12$ and $y = 5.$

Hence $\sqrt{17 + \sqrt{240}} = \sqrt{12} + \sqrt{5}.$

$$\sqrt{17 + \sqrt{240}} = \pm (2\sqrt{3} + \sqrt{5}).$$

If in (6) we select -7 as the square root of 49 , the values of x and y are interchanged, and the final answer is the same as before.

NOTE. The preceding example gives a rational x and y , because $17^2 - 240$ is a perfect square. In general the method can be used for the finding of $\sqrt{a + \sqrt{b}}$ only if $a^2 - b$ is a perfect square.

Ex. Find $\sqrt{2a - \sqrt{4a^2 - 4b^2}}.$

Let $\sqrt{2a - \sqrt{4a^2 - 4b^2}} = \sqrt{x} - \sqrt{y}.$

Squaring both members, $2a - \sqrt{4a^2 - 4b^2} = x - 2\sqrt{xy} + y.$

Then, by § 274, $x + y = 2a.$ (1)

$$2\sqrt{xy} = \sqrt{4a^2 - 4b^2}. \quad (2)$$

Squaring (1), $x^2 + 2xy + y^2 = 4a^2.$ (3)

Squaring (2), $4xy = 4a^2 - 4b^2.$ (4)

(3)-(4), $x^2 - 2xy + y^2 = 4b^2.$

Extracting the square root, $x - y = 2b.$

But $x + y = 2a.$

Therefore $x = a + b, y = a - b,$

and $\sqrt{2a - \sqrt{4a^2 - 4b^2}} = \pm (\sqrt{a + b} - \sqrt{a - b}).$

EXERCISE 105

Extract the square roots of the following binomials:

1. $7 + 2\sqrt{10}.$

5. $7 + \sqrt{40}.$

2. $3 + 2\sqrt{2}.$

6. $19 - 4\sqrt{12}.$

3. $6 + \sqrt{32}.$

7. $8 - \sqrt{55}.$

4. $7 + \sqrt{24}.$

8. $46 - 3\sqrt{20}.$

- | | |
|--------------------------|--|
| 9. $9 + 4\sqrt{5}$. | 14. $33 - 10\sqrt{8}$. |
| 10. $83 + 12\sqrt{35}$. | 15. $m + n + 2\sqrt{mn}$. |
| 11. $52 + 14\sqrt{3}$. | 16. $2a + b - 2\sqrt{a^2 + ab}$. |
| 12. $69 + 16\sqrt{5}$. | 17. $a^2 + 2a + 2\sqrt{a^3 - 1}$. |
| 13. $7 - \sqrt{33}$. | 18. $2 + 2a^2 - \sqrt{4(1 + a^2 + a^4)}$. |

276. To find the square root of a binomial square by inspection. According to § 65,

$$\begin{aligned}
 (\sqrt{5} + \sqrt{3})^2 &= 5 + 2\sqrt{5 \cdot 3} + 3. \\
 &= 8 + 2\sqrt{15}.
 \end{aligned}$$

If on the other hand we had to find $\sqrt{8 + 2\sqrt{15}}$, the problem would be quite simple, if presented in the form $\sqrt{5 + 2\sqrt{3 \cdot 5} + 3}$. To reduce it to this form, we must find two numbers whose sum is 8 and whose product is 15, viz. 5 and 3.

Ex. 1. Find $\sqrt{12 + 2\sqrt{20}}$.

Find two numbers whose sum is 12 and whose product is 20. These numbers are 10 and 2.

$$\begin{aligned}
 \sqrt{12 + 2\sqrt{20}} &= \sqrt{10 + 2\sqrt{10 \times 2} + 2}. \\
 &= \sqrt{10} + \sqrt{2}.
 \end{aligned}$$

Ex. 2. Find $\sqrt{11 - 6\sqrt{2}}$.

Write the binomial so that the coefficient of the irrational term is 2.

$$\sqrt{11 - 6\sqrt{2}} = \sqrt{11 - 2\sqrt{18}}.$$

Find two numbers whose sum is 11, and whose product is 18. The numbers are 9 and 2.

$$\begin{aligned}
 \text{Hence} \quad \sqrt{11 - 6\sqrt{2}} &= \sqrt{9 - 2\sqrt{2 \cdot 9} + 2}. \\
 &= \sqrt{9} - \sqrt{2}. \\
 &= 3 - \sqrt{2}.
 \end{aligned}$$

EXERCISE 106

Find by inspection the square roots of the following expressions :

- | | | |
|----------------------|-----------------------|-------------------------|
| 1. $3 - 2\sqrt{2}$. | 5. $7 - 2\sqrt{10}$. | 8. $11 - 4\sqrt{6}$. |
| 2. $5 - 2\sqrt{6}$. | 6. $7 - \sqrt{48}$. | 9. $6 + \sqrt{32}$. |
| 3. $4 - 2\sqrt{3}$. | 7. $11 - 4\sqrt{7}$. | 10. $30 + 12\sqrt{6}$. |
| 4. $6 + 2\sqrt{5}$. | | |

Find the fourth roots of:

- | | |
|-------------------------|-------------------------|
| 11. $17 - 12\sqrt{2}$. | 12. $28 + 16\sqrt{3}$. |
|-------------------------|-------------------------|

RADICAL EQUATIONS

277. A radical equation is an equation involving an irrational root of an unknown number.

$\sqrt{x} = 5$, $\sqrt[3]{x+3} = 7$, $(2x - x^2)^{\frac{1}{5}} = 1$, are radical equations.

278. Radical equations are rationalized, *i.e.* they are transformed into rational equations, by raising both members to equal powers.

Before performing the involution, it is necessary in most examples to simplify the equation as much as possible, and to transpose the terms so that one radical stands alone in one member.

If all radicals do not disappear through the first involution, the process must be repeated.

Ex. 1. Solve $\sqrt{x^2 + 12} - x = 2$.

Transposing x , $\sqrt{x^2 + 12} = x + 2$.

Squaring both members, $x^2 + 12 = x^2 + 4x + 4$.

Transposing and uniting, $-4x = -8$.

Dividing by -4 , $x = 2$.

Check. The value $x = 2$ reduces each member to 2.

Ex. 2. Solve $6\sqrt{x+5} - 3 = 4\sqrt{x+5} + 17$.

Transposing and uniting, $2\sqrt{x+5} = 20$.

Dividing by 2, $\sqrt{x+5} = 10$.

Squaring both members, $x+5 = 100$.

Transposing, $x = 95$.

Check. $6\sqrt{95+5} - 3 = 57$; $4\sqrt{95+5} + 17 = 57$.

Ex. 3. Solve $\sqrt{x+5} + \sqrt{x} - \sqrt{4x+9} = 0$.

Transpose $\sqrt{4x+9}$, $\sqrt{x+5} + \sqrt{x} = \sqrt{4x+9}$.

Squaring both members, $x+5 + 2\sqrt{x^2+5x} + x = 4x+9$.

Transposing and uniting, $2\sqrt{x^2+5x} = 2x+4$.

Dividing by 2, $\sqrt{x^2+5x} = x+2$.

Squaring both members, $x^2+5x = x^2+4x+4$.

Transposing and uniting, $x = 4$.

Check. $\sqrt{4+5} + \sqrt{4} - \sqrt{16+9} = 3 + 2 - 5 = 0$.

Ex. 4. Solve $\sqrt{14-x} + \sqrt{11-x} = \frac{3}{\sqrt{11-x}}$.

Clearing of fractions, $\sqrt{154-25x+x^2} + 11-x = 3$.

Transposing, $\sqrt{154-25x+x^2} = x-8$.

Squaring, $154-25x+x^2 = x^2-16x+64$.

Transposing and uniting, $-9x = -90$.

Dividing by -9 , $x = 10$.

Check. $\sqrt{14-10} + \sqrt{11-10} = 3$; $\frac{3}{\sqrt{11-10}} = 3$.

EXERCISE 107

Solve the following equations:

1. $\sqrt{x} = a$.

5. $5 - 3\sqrt{2x-1} = 2$.

2. $\sqrt{x} + 5 = 7$.

6. $\sqrt{x-a} - b = c$.

3. $5 + \sqrt[3]{x} = 8$.

7. $13 - \sqrt{4x^2 + 7x - 8} = 2x$.

4. $5 + 3\sqrt{x} = 7$.

8. $\sqrt{(x-5)(4x+4)} + 6 = 2x$.

$$9. 4x - \sqrt{(2x+5)(8x-7)} + 7 = 6.$$

$$10. \sqrt{x^2 + 12x} = x + 2.$$

$$11. \sqrt[5]{x} + 2 = 3.$$

$$12. \sqrt{x+7} - \sqrt{x-5} = 2.$$

$$13. \sqrt{3x+10} = 7 - \sqrt{3x-11}.$$

$$14. \sqrt{7x+1} = \sqrt{7x+18} - 1.$$

$$15. \sqrt{4x+1} + \sqrt{4x+25} = 12.$$

$$16. \sqrt{x-3} - 1 = \sqrt{x-10}.$$

$$17. \sqrt{5x+6} + 2 = \sqrt{5x+34}.$$

$$18. \sqrt{9x-11} + 3 = \sqrt{9x+28}.$$

$$19. \sqrt{x+60} = 2\sqrt{x+5} + \sqrt{x}.$$

$$20. \sqrt{x+9} - \sqrt{x+2} = \sqrt{4x-27}.$$

$$21. \sqrt{12x-11} - \sqrt{3x+1} = \sqrt{3x-6}.$$

$$22. \sqrt{9x-2} = \sqrt{25x-11} - \sqrt{4x-3}.$$

$$23. \sqrt{9x-5} - 2\sqrt{4x-15} - \sqrt{x-5} = 0.$$

$$24. \sqrt{6 + \sqrt{4 + \sqrt{x+2}}} = 3.$$

$$25. (2\sqrt{x}+3)(2\sqrt{x}-3) = 7.$$

$$26. (3\sqrt{x}+2)(3\sqrt{x}-2) = 5.$$

$$27. \frac{2\sqrt{x}+1}{3\sqrt{x}-2} = \frac{2\sqrt{x}+3}{3\sqrt{x}-5}.$$

$$31. \sqrt{x+4} = \frac{x+1}{\sqrt{x-1}}.$$

$$28. \frac{4\sqrt{x}-7}{5\sqrt{x}-6} = \frac{\sqrt{x}-7}{\sqrt{x}-6}.$$

$$32. \sqrt{9x+10} = \frac{6x+10}{\sqrt{4x+9}}.$$

$$29. \frac{11 - \sqrt{25x}}{27 - 5\sqrt{x}} = \frac{\sqrt{x}+2}{\sqrt{x}-4}.$$

$$33. \sqrt{2x-1} = \frac{2(x-3)}{\sqrt{2x-10}}.$$

$$30. \sqrt{7x+2} = \frac{5x+6}{\sqrt{7x+2}}.$$

$$34. \frac{6x-8\sqrt{x}+8}{4x-7\sqrt{x}+12} = \frac{2}{3}.$$

$$35. \sqrt{19-x} + \sqrt{11-x} = \frac{4}{\sqrt{11-x}}.$$

$$36. \sqrt{a-x} + \sqrt{b-x} = \frac{b}{\sqrt{b-x}}.$$

$$37. \sqrt{x+2} - \sqrt{x-9} = \sqrt{x-18} - \sqrt{x-25}.$$

$$38. \sqrt{x-7} - \sqrt{x-10} = \sqrt{x+5} - \sqrt{x-2}.$$

$$39. \frac{2\sqrt{ax}-b}{2\sqrt{ax}+b} = \frac{3\sqrt{ax}-2b}{3\sqrt{ax}+4b}.* \quad 40. \frac{3\sqrt{3x}+5}{3\sqrt{3x}-5} = \frac{7}{2}.$$

$$41. \sqrt{35+7\sqrt{5x+4}} = 14.$$

$$42. (1-\sqrt{x}) : (1+3\sqrt{x}) = 1 : 4.$$

$$43. (4+\sqrt{x}) : (4-\sqrt{x}) = 4 : 1.$$

$$44. (a+\sqrt{x}) : (a-\sqrt{x}) = a+b : a-b.$$

$$45. 6 + \sqrt{\frac{3a + \sqrt{x-b}}{a}} = 8. \quad 46. \begin{cases} 3\sqrt{x} + 5\sqrt{y} = 45. \\ 7\sqrt{x} - 4\sqrt{y} = 11. \end{cases}$$

$$47. \sqrt{3x} - \sqrt{2x} + \sqrt{x} = -2\sqrt{2}.$$

$$48. \begin{cases} 6\sqrt{x-5} - 5\sqrt{y+4} = 4. \\ 13\sqrt{x-5} + 2\sqrt{y+4} = 60. \end{cases}$$

EXTRANEIOUS ROOTS AND EQUIVALENT EQUATIONS

279. Solve the equation :

$$5 + \sqrt{x} = 2 \quad (1)$$

$$\text{Transposing,} \quad \sqrt{x} = -3 \quad (2)$$

$$\text{Squaring both members,} \quad x = 9. \quad (3)$$

But the root $x=9$ does not satisfy the equation (1) since,

$$5 + \sqrt{9} = 8.$$

* Consider the equation a proportion and apply composition and division.

If the value of $\sqrt{9}$ had not been restricted (§ 255) to its positive value, 9 would be the root. On the other hand, no positive value of \sqrt{x} can satisfy the equation, since 5 plus a positive number cannot equal 2.

Consequently equation (1) has no root, although the student should bear in mind that this is the consequence of the arbitrary exclusion of negative values of roots.

280. If $x=9$ is not a root of the equation $5 + \sqrt{x} = 2$, it becomes necessary to ascertain why the solution produced this value.

In solving an equation, we usually proceed as if the given equation were true and we had to prove the correctness of each following one; or we prove that (3) is true if (1) is true. The real problem, however, is the opposite one, viz. to prove equation (1) is true if equation (3) is true. That is, we should start from equation (3) and prove successively (2) and (1).

But the members of (2) are the square roots of the members of (3), and if two quantities are equal, their square roots are not necessarily equal, as shown by the following illustration: $(-3)^2 = (+3)^2$, while -3 does not equal $+3$. Hence (2) does not need to be true, if (3) is true.

In general, squaring the two members of an equation introduces a new root, as can be seen from the following example:

Let $x = a$.

Squaring both members, $x^2 = a^2$.

The roots of the second equation are $+a$ and $-a$, while the first one has only one root, $+a$.

281. Equivalent equations are equations which have the same roots; as $x + 4 = \sqrt{x}$ and $x = \sqrt{x} - 4$.

282. A new root which is introduced by performing the same operations on both members of an equation is called an **extraneous root**.

283. Squaring both members of an equation frequently introduces an extraneous root.

284. Multiplying both members of an equation by an expression involving x usually introduces an extraneous root.

E.g. $x-4=0$ has one root, $x=4$.

Multiplying both members by x , we obtain

$$x^2 - 4x = 0,$$

an equation which has two roots, 4 and 0.

285. *The results of a radical equation must be substituted in the given equation to determine whether the roots are true roots or extraneous roots.*

EXERCISE 108

Solve the following equations, and if the resulting roots are extraneous, change the equations so as to make the answer true roots:

1. $5 - \sqrt{x+4} = 6.$

2. $\sqrt{x+5} - \sqrt{x} = \sqrt{4x+9}.$

3. $\sqrt{x+9} - \sqrt{x+2} = \sqrt{4x-27}.$

4. $\sqrt{x+7} + \sqrt{x-5} = 2.$

5. $\sqrt{x+3} - \sqrt{x-2} = 5.$

6. $\sqrt{5+x} + \sqrt{x-2} = 7.$

CHAPTER XVI

THE FACTOR THEOREM

286. If $x^3 - 3x^2 + 4x + 8$ is divided by $x - 2$ and there is a remainder (which does not contain x), then

$$x^3 - 3x^2 + 4x + 8 = (x - 2) \times \text{Quotient} + \text{Remainder}.$$

Or, substituting Q and R respectively for "Quotient" and "Remainder," and transposing,

$$R = x^3 - 3x^2 + 4x + 8 - (x - 2)Q.$$

As R does not contain x , we could, if Q was known, assign to x any value whatsoever and would always obtain the same answer for R .

If, however, we make $x = 2$, then $(x - 2)Q = 0$, no matter what the value of Q . Hence, even if Q is unknown, we can find the value of R by making $x = 2$.

$$R = 2^3 - 3 \cdot 2^2 + 4 \cdot 2 + 8 - 0 = 12.$$

Ex. 1. Without actual division, find the remainder obtained by dividing $3x^4 + 2x - 5$ by $x - 3$.

$$R = 3x^4 + 2x - 5 - (x - 3)Q.$$

Let

$$x = 3,$$

then

$$R = 3 \cdot 81 + 2 \cdot 3 - 5 - 0 = 244.$$

Ex. 2. Without actual division, find the remainder when $ax^4 + bx^3 + cx^2 + dx + e$ is divided by $x - m$.

$$R = ax^4 + bx^3 + cx^2 + dx + e - (x - m)Q.$$

Let

$$x = m,$$

then

$$R = am^4 + bm^3 + cm^2 + dm + e.$$

287. The Remainder Theorem. *If an integral rational expression involving x is divided by $x - m$, the remainder is obtained by substituting in the given expression m in place of x .*

E.g. The remainder of the division

$$(4x^5 - 4x + 11) \div (x + 3) \text{ is } 4(-3)^5 - 4(-3) + 11 = -949.$$

The remainder obtained by dividing

$$(x + 4)^4 - (x + 2)(x - 1) + 7 \text{ by } x - 1 \text{ is } 5^4 - 3 \cdot 0 + 7 = 632.$$

EXERCISE 109

Without actual division, find the remainder obtained by dividing:

1. $x^3 + 3x^2 - 3x + 2$ by $x - 4$.
2. $x^{10} - 7x^8 + 4x^5 - 2x - 4$ by $x - 1$.
3. $a^6 + 2a^2 - 4a + 1$ by $a - 3$.
4. $a^4 + 5a^2 + 2a + 1$ by $a + 2$.
5. $(x + 1)^3 - 3(x - 2)(x + 2) + (x + 1)^2$ by $x - 2$.
6. $x^3 - 4x^2m + 4xm^2 + m^3$ by $x - m$.
7. $x^3 - 2x^2n + 4xn^2 - n^3$ by $x + 2n$.
8. $(x + 4)^2 + (x + 3)^3 + (x + 2)^4$ by $x + 1$.
9. $(a - 1)(a - 3) - 4[(a - 3)(a - 4) - 3]$ by $a - 3$.
10. $x^5 + b^5$ by $x - b$.
11. $a^5 + b^5$ by $a + b$.
12. $a^{20} + b^{20}$ by $a + b$.

288. If the remainder is zero, the divisor is a factor of the dividend.

The Factor Theorem. *If a rational integral expression involving x becomes zero when m is written in place of x , $x - m$ is a factor of the expression.*

E.g. if $x^3 - 3x^2 - 2x - 8$ is divided by $x - 4$, the remainder equals $4^3 - 3 \cdot 4^2 - 2 \cdot 4 - 8 = 0$, hence $(x - 4)$ is a factor of $x^3 - 3x^2 - 2x - 8$.

289. A rational integral expression, $ax^n + bx^{n-1} \dots ex + f$, is divisible by $x - m$ only, if m is a factor of f . (§ 131.)

Hence, to factor the expression, we substitute for m exact divisors of f , and determine by means of the factor theorem whether $x - m$ is a factor or not.

Ex. 1. Factor $x^3 - 7x^2 + 7x + 15$.

The exact divisors of 15 are $+1, -1, +3, -3, +5, -5, +15, -15$.

Let $x = 1$, then $x^3 - 7x^2 + 7x + 15$ does not vanish.

Let $x = -1$, then $x^3 - 7x^2 + 7x + 15 = 0$.

Therefore $x - (-1)$, or $x + 1$, is a factor.

By dividing by $x + 1$, we obtain

$$\begin{aligned} x^3 - 7x^2 + 7x + 15 &= (x + 1)(x^2 - 8x + 15). \\ &= (x + 1)(x - 3)(x - 5). \end{aligned}$$

Ex. 2. Factor $x^3 + bx^2 - a^2x - a^2b$.

The exact divisors of a^2b are $-a, +a, -b, +b, -ab$, etc.

The substitution $x = a$ makes the expression vanish.

Dividing by $x - a$, we have

$$\begin{aligned} x^3 + bx^2 - a^2x - a^2b &= (x - a)(x^2 + bx + ax + ab). \\ &= (x - a)(x + a)(x + b). \end{aligned}$$

EXERCISE 110

Without actual division, show that:

1. $10x^3 - 4x^2 - 13x + 7$ is divisible by $x - 1$.
2. $a^4 - 4a^2 - 7a - 24$ is divisible by $a - 3$.
3. $a^4 + a^3 - ab^3 - b^3$ is divisible by $a - b$.
4. $x^3 + 3x^2 - 4x - 12$ is divisible by $x - 2$ and $x + 2$.
5. $(x + 1)^2(x - 2) - 4(x - 1)(x - 3) + 4$ is divisible by $x - 1$.
6. $6x[4(x + 1)(x + 2) - 47] - x^3 - x^2 + 3x - 6$ is divisible by $x - 2$.
7. $6x^5 - 3x^4 - 5x^3 + 5x^2 - 2x - 3$ is divisible by $x + 1$.

Resolve into factors:

8. $a^3 - 2a - 4$.
 9. $x^3 + 2x - 3$.
 10. $3x^3 - 8x + 5$.
 11. $a^3 + 6a^2 + 11a + 6$.
 12. $x^3 - 7x + 6$.
 13. $n^3 - 7n - 6$.
 14. $p^3 + 7p^2 + 14p + 8$.
 15. $m^3 - 19m + 30$.
 16. $4x^3 - 13x + 6$.
 17. $m^4 - 3m^3n - 5mn^3 + 18n^4$.
 18. $2x^3 - 5x^2 - 13x + 30$.
 19. $x^4 - x^3 - 7x^2 + x + 6$.
 20. $6x^3 + 7x^2 - x - 2$.
 21. $x^3 - (3a - b)x^2 + (2a^2 - 3ab)x + 2a^2b$.
 22. $x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc$.
 23. $x^3 - (3a + 2b)x^2 + (6ab + 2a^2)x - 4a^2b$.
 24. Find the H. C. F. of $3a^3 + 5a^2 - a + 2$ and $a^3 + a^2 - a + 2$.
 25. Find the H. C. F. of $9x^3 + 18x^2 - x - 10$ and $3x^3 + 13x^2 + 2x - 8$.
 26. Find the H. C. F. of $x^3 - x^2 - 5x - 3$ and $x^3 - 4x^2 - 11x - 6$.
-

290. If n is a positive integer, it follows from the Factor Theorem that

1. $x^n - y^n$ is always divisible by $x - y$.

For substituting y for x , $x^n - y^n = y^n - y^n = 0$.

2. $x^n + y^n$ is divisible by $x + y$, if n is odd.

For $(-y)^n + y^n = 0$, if n is odd.

By actual division we obtain the other factors, and have for any positive integral value of n ,

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots + y^{n-1}).^*$$

* The symbol ... means "and so forth to."

If n is odd,

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 \cdots + y^{n-1}).$$

$$\text{e.g. } x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4).$$

$$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).$$

291. It can readily be seen that $x^n + y^n$ is not divisible by either $x + y$ or $x - y$, if n is even, and that $x^n - y^n$ cannot be divided by $x + y$, if n is odd.

EXERCISE 111

State whether the following expressions are prime or not, and factor whenever possible.

$$1. \ a^5 - 1. \quad 4. \ m^5 + n^5. \quad 7. \ 1 + a^7. \quad 10. \ a^4 - b^{12}.$$

$$2. \ a^5 + 1. \quad 5. \ m^5 - n^5. \quad 8. \ 1 - a^9. \quad 11. \ b^9 - a^9.$$

$$3. \ a^2 + b^2. \quad 6. \ a^7 - 1. \quad 9. \ a^{11} - b^{11}. \quad 12. \ a^{10} - b^5.$$

[For additional examples of this type, see Appendix II.]

CHAPTER XVII *

GRAPHIC REPRESENTATION OF FUNCTIONS AND EQUATIONS

REPRESENTATION OF FUNCTIONS OF ONE VARIABLE

292. An expression involving one or several letters is called a **function** of these letters.

$x^2 - x + 7$ is a function of x .

$2xy - y^2 + 3y^3$ is a function of x and y .

293. If the value of a quantity changes, the value of a function of this quantity will change, *e.g.* if x assumes successively the values 1, 2, 3, 4, $x^2 - x + 7$ will respectively assume the values 7, 9, 13, 19. If x increases gradually from 1 to 2, $x^2 - x + 7$ will change gradually from 7 to 9.

294. A **variable** is a quantity whose value changes in the same discussion.

295. A **constant** is a quantity whose value does not change in the same discussion.

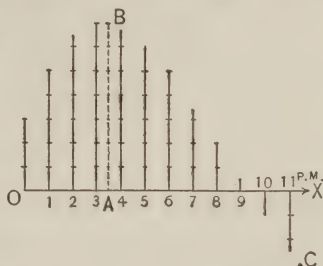
In the example of the preceding article, x is supposed to change, hence it is a variable, while 7 is a constant.

296. A convenient method for the representation of the various values of a function of a letter, when this letter changes, is the method of representing these values graphically; that is, by a diagram. This method is frequently used to represent in a concise manner a great many data referring to facts taken from physics, chemistry, technology, economics, etc.

* This chapter may be omitted on a first reading.

297. To give first a diagram of one of these applications, let us suppose that we have measured the temperatures at all hours from 12 M. to 11 P.M. on a certain day, and that we wish to represent the results graphically. Draw a line OX , and lay off on it equal parts, representing the hours. Then O represents 12 o'clock; 4, four o'clock, etc. At each point draw a perpendicular line, and make it equal to the temperature of the corresponding hour (any convenient length being taken as a unit).

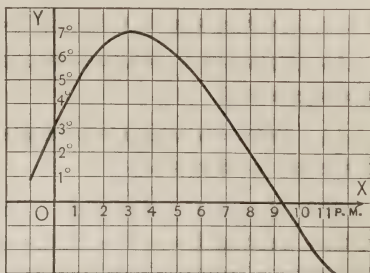
By inspection of the diagram it can be seen that the temperature at 12 o'clock was 3° , at 1 P.M. 5° , at 2 P.M. $6\frac{1}{2}^\circ$, etc.



But it is possible to represent in the diagram the temperatures of any particular time between 12 M. and 11 P.M.; thus the perpendicular AB indicates that the temperature at 3.30 was 7° . We may also omit the perpendicular and simply draw its end point; as point C . By measuring the distance of C from OX , we find that the temperature at 11.20 was -3° .

298. If we would represent the temperatures of every moment between 12 and 11.20, we would obtain an uninterrupted sequence of points, or a curved line, as shown in the next diagram. This curve is said to be a **graphical representation** or a **graph** of the temperatures from 12 to 11.20.

To find from the graph the temperatures at any hour, e.g. 2.30, take a point A , $2\frac{1}{2}$ units from O , and measure the length of the perpendicular at A (not drawn in the diagram).



EXERCISE 112

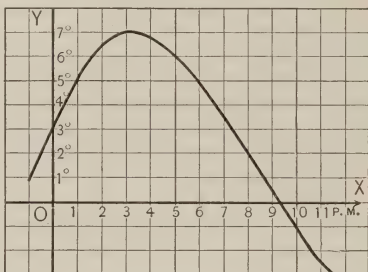
From the diagram find approximate answers to the following questions :

1. Determine the temperature at:

(a) 5 P.M. (b) 1.30 P.M.
(c) 5.45 P.M. (d) 11.45 A.M.

2. At what hour or hours was the temperature (a) 6° , (b) 5° , (c) 1° , (d) -1° , (e) 0° ?

3. At what hour was the temperature highest?



4. What was the highest temperature?

5. During what hours was the temperature above 5° ?

6. During what hours was the temperature between 3° and 4° ?

7. During what hours was the temperature above 0° ?

8. During what hours was the temperature below 0° ?

9. How much higher was the temperature at 4 than at 8 P.M.?

10. At what hour was the temperature the same as at 1 P.M.?

11. During what hours did the temperature increase?

12. During what hours did the temperature decrease?

13. Between which two successive hours did the temperature change least?

14. Between which two successive hours did the temperature increase most rapidly?

299. Graph of a function. To represent the various values which an algebraic function, *e.g.* $3 + \frac{3}{2}x - \frac{1}{4}x^2$, assumes, when x changes from -3 to $+9$, let us substitute in $3 + \frac{3}{2}x - \frac{1}{4}x^2$, successively the values $-3, -2, -1, 0, 1$, etc.

If x respectively $= -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9$,
then $3 + \frac{3}{2}x - \frac{1}{4}x^2 = -3\frac{3}{4}, -1, 1\frac{1}{4}, 3, 4\frac{1}{4}, 5, 5\frac{1}{4}, 5, 4\frac{1}{4}, 3, 1\frac{1}{4}, -1, -3\frac{3}{4}$.

Selecting any convenient length as a unit, we lay off on a line OX from O the values of x , and at each point erect a perpendicular equal to the corresponding value of $3 + \frac{3}{2}x - \frac{1}{4}x^2$.

To find *e.g.* the point representing that,

for $x = 2, 3 + \frac{3}{2}x - \frac{1}{4}x^2 = 5$,

we lay off $OA = 2$, and at A draw a perpendicular $AB = 5$. Similarly for all other points, as C, D , etc. Since the positive

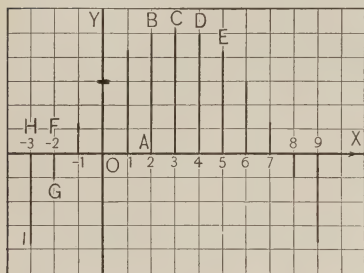
values of x are laid off from O toward X , negative values of x must be laid off from O in the opposite direction; as $OF = -2$. Similarly negative values of $3 + \frac{3}{2}x - \frac{1}{4}x^2$ must be laid off in a direction opposite to the positive values, as $FG = -1$, or $HI = -3\frac{3}{4}$.

300. The lines OA and AB are called the **coördinates** of point B , OA is the **abscissa**, AB the **ordinate** of point B . The abscissa of I is OH , its ordinate HI .

301. The line OX is called the **axis of abscissas** or **x -axis**. The perpendicular OY , erected at O , is the **axis of ordinates** or **y -axis**. The point O is called the **origin**. Abscissas measured to the right of the origin, and ordinates above the x -axis, are considered positive.

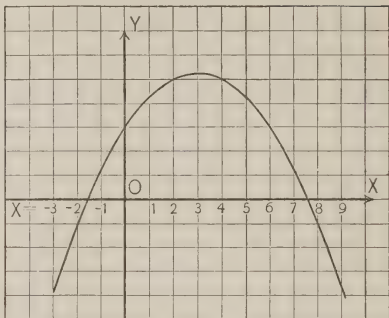
302. The point whose abscissa is x and whose ordinate is y is often represented by (x, y) . Thus $(2, 5)$ represents the point B , $(-2, -1)$ is the point G .

The point (x, y) can also be obtained by first laying off y on the y -axis, and then drawing x perpendicular to oy at the extremity of y .



303. If we should represent the values of the function $3 + \frac{3}{2}x - \frac{1}{4}x^2$ for all values of x from $x = -3$ to $x = +9$, and should omit the perpendiculars, we should obtain a continuous succession of points, or a curved line as shown in the diagram.

This curve is called the **graph of the function**.



NOTE. It is convenient to use for such drawings coördinate paper, *i.e.* paper divided into squares, as shown in the diagram. An ordinary ruled sheet can also be used to advantage.

EXERCISE 113

Find from the diagram of the preceding article approximate answers to Exs. 1-19.

1. Determine the value of the function, $3 + \frac{3}{2}x - \frac{1}{4}x^2$, if (a) $x = -2\frac{1}{2}$, (b) $x = 5$, (c) $x = 6\frac{1}{2}$, (d) $x = 8\frac{1}{2}$.
2. What value or values of x will make $3 + \frac{3}{2}x - \frac{1}{4}x^2$ equal to (a) 3, (b) 4, (c) -2 , (d) 1, (e) 0?
3. What is the greatest value which $3 + \frac{3}{2}x - \frac{1}{4}x^2$ can assume between $x = -3$ and $x = +9$?
4. What value of x produces the greatest value of $3 + \frac{3}{2}x - \frac{1}{4}x^2$?
5. What values of x produce positive values of $3 + \frac{3}{2}x - \frac{1}{4}x^2$?
6. What values of x produce negative values of the same function?
7. Between what values must x lie to produce a function $3 + \frac{3}{2}x - \frac{1}{4}x^2$ greater than $+3$?

8. What values of x produce values of the same function smaller than -1 ?

9. What values of x make the same function equal to zero?

10. What then are the roots of the equations $3 + \frac{3}{2}x - \frac{1}{4}x^2 = 0$?

11. Find two roots of the equation $3 + \frac{3}{2}x - \frac{1}{4}x^2 = 2$.

12. Find two roots of the equation $3 + \frac{3}{2}x - \frac{1}{4}x^2 = 3$.

13. Find two roots of the equation $3 + \frac{3}{2}x - \frac{1}{4}x^2 = 4$.

14. Has the equation $3 + \frac{3}{2}x - \frac{1}{4}x^2 = 6$ any real roots?

15. Find the value of m for which the equation $3 + \frac{3}{2}x - \frac{1}{4}x^2 = m$ has only one root.

16. How much smaller is the value of the function for $x=6$ than for $x=3$?

17. Which other value of x produces the same value of the function as $x=6$?

18. If x increases from -3 to $+2$, does the function also increase?

19. Up to what value of x will the function increase when x increases?

20. Locate the points $(2, 5)$, $(3, 6)$, $(3, -2)$, $(0, 5)$, $(-2, 5)$, $(-5, -4)$, $(-3\frac{1}{2}, 0)$, $(0, -\frac{1}{2})$, $(-1, 0)$.

21. Locate the points $(-2, 0)$, $(0, -2)$, $(-\frac{1}{2}, -\frac{1}{2})$.

22. Locate the points $(-3, 3)$, $(-2, 3)$, $(-1, 3)$, $(0, 3)$, $(1, 3)$.

23. Where do all points lie whose ordinate is 3?

24. Locate the points representing the values of $\frac{2}{3}x + 1$, if $x = -3, -2, -1, 0, 1, 2, 3$.

25. Construct the graph of the function $\frac{2}{3}x + 1$, from $x = -3$, to $x = +3$.

26. From the graph constructed in Ex. 25 find approximately:

(a) the value of $\frac{2}{3}x + 1$ if $x = -2\frac{1}{2}, -\frac{1}{2}, 1\frac{1}{2}$.

(b) the value of x , if $\frac{2}{3}x + 1 = -\frac{1}{2}, 2, 2\frac{1}{2}$.

- (c) the values of x which make $\frac{2}{3}x + 1$ positive.
- (d) the values of x which make $\frac{2}{3}x + 1$ negative.
- (e) the value of x that makes $\frac{2}{3}x + 1 = 0$.
- (f) the root of the equation $\frac{2}{3}x + 1 = 0$.
- (g) the root of the equation $\frac{2}{3}x + 1 = 2$.

27. Construct the graph of x^2 , from $x = -5$, to $x = +5$, and from the diagram determine the following values: $(-1\frac{1}{2})^2$, $(3\frac{1}{4})^2$, $(2.2)^2$, $\sqrt{7}$, $\sqrt{12}$, $\sqrt{4.5}$. (Make the scale unit of the x equal to 10 times the scale unit of the x^2 .)

GRAPHIC SOLUTION OF EQUATIONS INVOLVING ONE UNKNOWN QUANTITY

304. A rational integral equation which contains the n th power of the unknown quantity, but no higher power, is called an equation of the n th degree.

$x^3 - 2x - 4$ is an equation of the third degree.

305. The roots of equations of the first and higher degrees can be found approximately by the graphical method.

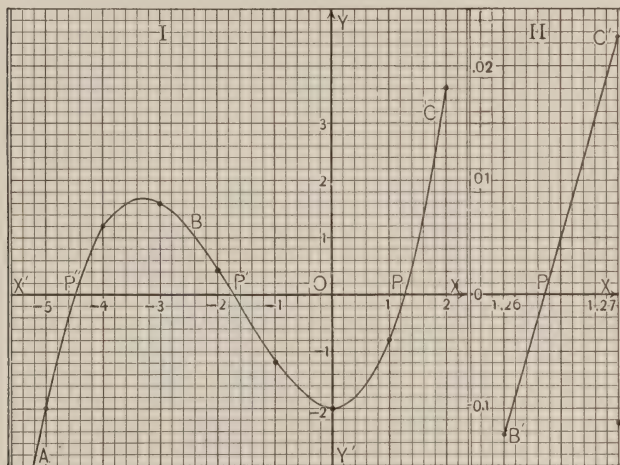
Ex. Find graphically the roots of the equation $\frac{1}{5}x^3 + x^2 - 2 = 0$.

To obtain the values of the function for the various values of x , the following arrangement may be found convenient:

(Compute each column before commencing the next, and see table on p. 288.)

x	x^2	x^3	$\frac{1}{5}x^3$	$\frac{1}{5}x^3 + x^2$	$\frac{1}{5}x^3 + x^2 - 2$
-5	25	-125	-25	0	-2
-4	16	-64	-12.8	3.2	1.2
-3	9	-27	-5.4	3.6	1.6
-2	4	-8	-1.6	2.4	.4
-1	1	-1	-.2	.8	-1.2
0	0	0	0	0	-2
1	1	1	.2	1.2	-.8
2	4	8	1.6	5.6	3.6
3	9	27	5.4	14.4	12.4

Locating the points $(-5, -2)$, $(-4, 1.2)$, etc., and joining, produces the graph ABC . Since ABC intersects the x -axis at three points, P , P' , and P'' , three values of x make the function zero. Hence there are three roots which are found, by measurement of OP'' , OP' , and OP , to be approximately -4.5 , -1.7 , and 1.25 .



To find a more exact answer for one of these roots, *e.g.* OP , we draw the portion of the diagram which contains P on a larger scale.

If $x = 1.25$, the function equals $-.0469$, *i.e.* it is negative. Hence it appears from the diagram that the root must be larger, and we substitute $x = 1.26$, which produces $\frac{1}{5}x^3 + x^2 - 2 = -.0123$. This again being a negative value, we substitute $x = 1.27$, which produces the positive value, $.0226$. The root, therefore, must lie between 1.26 and 1.27 .

Making the side of each small square (diagram II) equal to $.001$, we locate the points $(1.26, -.0123)$ and $(1.27, .0226)$, *i.e.* B' and C' . Since in nearly all cases small portions of the curve are almost straight lines, we join the two points by a straight line $B'C'$, which intersects the x -axis in P .

The measurement of OP gives the root

$$x = 1.2637,$$

an answer whose last place is not quite reliable. Restricting the answer to three places, we obtain the root

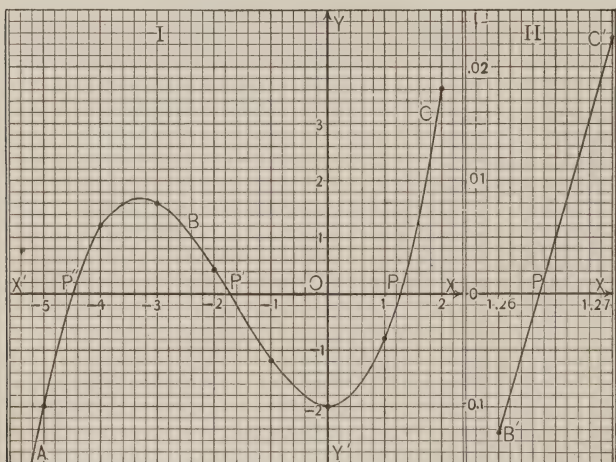
$$x = 1.264.$$

If a greater degree of accuracy is required, a third drawing on a still greater scale must be constructed.

EXERCISE 114

1. From the diagram of the preceding example find approximate answers to the following questions:

(a) What is the value of $\frac{1}{5}x^3 + x^2 - 2$ if $x = -1.2$? if $x = -3.4$? if $x = 1.5$?



(b) What values of x will make $\frac{1}{5}x^3 + x^2 - 2$ equal to 1? equal to 1.4? equal to -2 ?

(c) What is the greatest value which $\frac{1}{5}x^3 + x^2 - 2$ can assume for negative values of x ?

(d) What negative value of x produces the greatest value of the function?

(e) What value of x between -3 and 2 produces the smallest value of the function?

(f) What is the smallest value of the function between $x = -3$ and $x = +3$?

(g) Between what values must x lie to produce (1) a positive value of the function; (2) a negative value of the function?

(h) Find the roots of the equation:

$$\frac{1}{5}x^3 + x^2 - 2 = -1.$$

(i) Find the roots of the equation:

$$\frac{1}{5}x^3 + x^2 - 2 = 1.2.$$

(k) How many roots has the equation:

$$\frac{1}{5}x^3 + x^2 - 2 = 3?$$

Find the value of this root, or these roots.

(l) How many roots has the equation:

$$\frac{1}{5}x^3 + x^2 - 2 = -2?$$

(m) Between what values must m lie in order to give the equation $\frac{1}{5}x^3 + x^2 - 2 = m$, three roots?

(n) In the same equation, what values of m produce two roots?

(o) In the same equation, between what limits must m lie to produce only one root?

(p) If x increases, between what values of x does the function (a) increase, (b) decrease?

2. Construct the graph of $2 + x - \frac{1}{6}x^2$ from $x = -2$ to $x = 9$, and from the diagram determine approximately:

(a) the value of the function if $x = 2\frac{1}{2}$, if $x = -1\frac{1}{2}$.

(b) the value or values of x if the function equals -1 , $+2$.

(c) the values of x that make the function positive.

(d) the values of x that make the function negative.

(e) the root or roots of the equation obtained by making the function equal to zero.

(*f*) the root or roots of the equation obtained by making the function equal to 1.

(*g*) the greatest value of the function between the given points.

(*h*) the value of x that produces the greatest value of the function.

In each of the Exs. 3-9 construct the graph of the function, and from the diagram determine, whenever possible, the answers to questions (*a*), (*b*), (*c*), (*d*), (*e*), (*f*), and (*g*), given in Ex. 2.

3. $3 - \frac{3}{2}x$, from $x = -2$, to $x = 4$.
4. $x + 1$, from $x = -4$, to $x = +4$.
5. $\frac{1}{2}x^2 - x - 2$, from $x = -3$, to $x = +5$.
6. $\frac{1}{8}x^2$, from $x = -6$, to $x = 8$.
7. $\frac{6}{x} - \frac{3}{2}$, from $x = 1$, to $x = 12$.
8. $\frac{1}{10}x^3 - x + 1$, from $x = -4$, to $x = 4$.
9. $\sqrt{25 - x^2}$, from $x = -5$, to $x = 5$.

306. It can be proved that the **graphs of the functions of the first degree** involving one unknown quantity, are straight lines, hence two points are sufficient for the construction of these graphs. (This is true whether the scales of the abscissas and ordinates are equal or unequal.)

10. Draw the graph of $4x - 5$.

11. Draw the graph of $3 - 2x$.

12. Degrees of the Fahrenheit scale are expressed in degrees of the Centigrade scale by the formula $C = \frac{5}{9}(F - 32)$.

(*a*) Draw the graph of $\frac{5}{9}(F - 32)$, from $F = -5$, to $F = 40$.

(*b*) From the diagram find the number of degrees of Centigrade equal to $-1^\circ F.$, $9^\circ F.$, $14^\circ F.$, $32^\circ F.$

(*c*) Change to Fahrenheit readings $-10^\circ C.$, $0^\circ C.$, $1^\circ C.$

13. The formula for the distance traveled by a falling body is $S = \frac{1}{2} gt^2$.

(a) Represent $\frac{1}{2} gt^2$ graphically from $t = 0$ to $t = 5$. (Assume $g = 10$ meters, and make the scale unit of the t equal to 10 times the scale unit of the $\frac{1}{2} gt^2$.)

(b) How far does a body fall in $2\frac{1}{2}$ seconds?

(c) In how many seconds does a body fall 30 meters?

Solve graphically:

14. $4x + 3 = 0$.

15. $6x - 5 = 0$.

Solve the following equations by the graphical method, and find the greatest or the smallest value of the function:

(For the squares and the cubes of numbers, see table on p. 288.)

16. $x^2 - 3x - 3 = 0$.*

24. $x^3 + x - 3 = 0$.

17. $x^2 - x - \frac{15}{4} = 0$.

25. $x^3 - 4x + 1 = 0$.

18. $x^2 + \frac{1}{3}x + \frac{20}{9} = 0$.

26. $x^3 - 2x^2 - 5x + 5 = 0$.†

19. $x^2 - 2x - 9 = 0$.

27. $x^4 - 10x^2 + 8 = 0$.‡

20. $x^2 + \frac{14}{5}x - 11 = 0$.

28. $x^4 - 17x^2 + x + 54 = 0$.

21. $\frac{5}{8}x^2 - x - \frac{21}{8} = 0$.

29. $2^x + x - 4 = 0$.

22. $6x^2 - 13x + 4 = 0$.

30. $\frac{6}{x} - 2^x = 0$.

23. $\frac{1}{3}x^3 + x - 4 = 0$.

* A more convenient method for solving, graphically, equations of the second degree is given in Chapter XVIII.

† To avoid large ordinates, make the scale unit of ordinates $\frac{1}{5}$ of the scale unit of the abscissas. The same graph may be obtained by dividing each term of the equation by 5 and using equal scale units for ordinates and abscissas.

‡ Make the scale unit of the ordinates $\frac{1}{10}$ of the scale unit of the abscissas.

GRAPHIC SOLUTION OF EQUATIONS INVOLVING TWO UNKNOWN QUANTITIES

307. In § 303 the graph of the function $3 + \frac{3}{2}x - \frac{1}{4}x^2$ was constructed and discussed. If $3 + \frac{3}{2}x - \frac{1}{4}x^2$ be denoted by y , then the ordinate represents the various values of y , and the diagram (p. 276) represents the equation,

$$y = 3 + \frac{3}{2}x - \frac{1}{4}x^2. \quad (1)$$

The coördinates of every point of the curve satisfy equation (1), and every set of real values of x and y satisfying equation (1) is represented by a point in the curve.

308. The curve representing an equation is called the **graph** or **locus** of the equation.

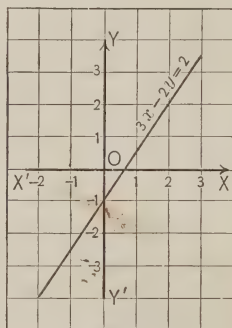
309. If an equation containing two unknown quantities can be reduced to the form $y = f(x)$, when $f(x)$ represents a function of x , then the equation can be represented graphically.

Ex. 1. Represent graphically $3x - 2y = 2$.

Solving for y ,
$$y = \frac{3x - 2}{2}.$$

Hence, if x equals $-2, -1, 0, 1, 2, 3$;
then y equals $-4, -2\frac{1}{2}, -1, \frac{1}{2}, 2, 3\frac{1}{2}$.

Locating the points $(-2, -4), (-1, -2\frac{1}{2})$, etc., and drawing a line through them, we obtain the graph of the equation, which is a straight line.



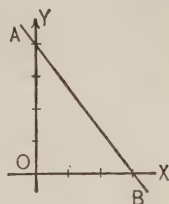
310. The graph of an equation of the first degree involving two unknown quantities is always a straight line, and hence it can be constructed if two points are located. (§ 306.)

Ex. 2. Draw the locus of $4x + 3y = 12$.

If $x = 0, y = 4$; if $y = 0, x = 3$.

Hence, locate points $(0, 4)$ and $(3, 0)$, and join them by a straight line AB . AB is the required graph.

NOTE. Equations of the first degree are called *linear* equations, because their graphs are straight lines.



EXERCISE 115

Draw the loci of the following equations:

- | | | |
|---------------------|--------------------|------------------------------|
| 1. $x + y = 4$. | 5. $x + y = -10$. | 9. $12x + 15y = 48$. |
| 2. $x - 2y = 4$. | 6. $y = -4$. | 10. $y - \sqrt{x} + 2 = 0$. |
| 3. $2x - 3y = 12$. | 7. $x + y = 0$. | 11. $(1.1)^x - y = 0$. |
| 4. $x - y = 0$. | 8. $y = 2x$. | 12. $xy + y - x^2 = 0$. |

311. Graphical solution of a linear system.

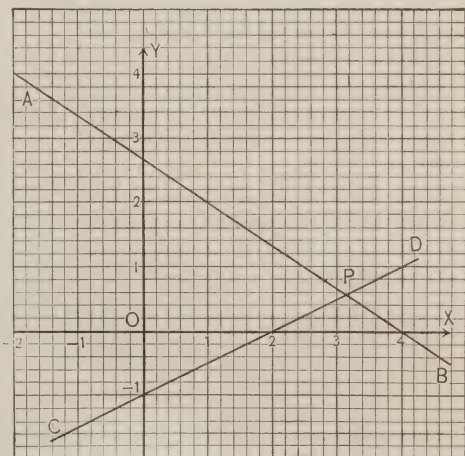
To find the roots of the system.

$$2x + 3y = 8, \quad (1)$$

$$x - 2y = 2. \quad (2)$$

By the method of the preceding article construct the graphs AB and CD of (1) and (2) respectively. The coördinates of every point in AB satisfy the equation (1), but only one point in AB also satisfies equation (2), viz. P , the point of intersection of AB and CD .

By measuring the coördinate of P , we obtain the roots, $x = 3.15, y = .57$.



312. The roots of two simultaneous equations are represented by the coördinates of the point (or points) at which their graphs intersect.

313. Since two straight lines which are not coincident nor parallel have only one point of intersection, simultaneous linear equations have only one pair of roots.

314. Equations of higher degree, however, can have several points of intersection, and hence several pairs of roots.

Ex. 1. Solve graphically the following system:

$$\begin{cases} x^2 + y^2 = 25, & (2) \\ 3x - 2y = -6. & (1) \end{cases}$$

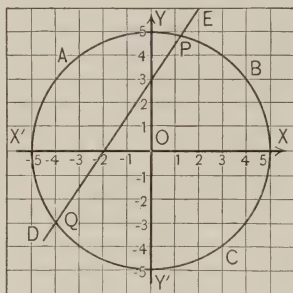
Solving (1) for y , $y = \sqrt{25 - x^2}$.

Therefore, if x equals $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$, y equals respectively $0, \pm 3, \pm 4, \pm 4.5, \pm 4.9, \pm 5, \pm 4.9, \pm 4.5, \pm 4, \pm 3, 0$.

Locating the points $(-5, 0), (-4, +3), (-4, -3)$, etc., and joining, we obtain the graph (a circle) ABC of the equation $x^2 + y^2 = 25$.

Locating two points of equation (2), e.g. $(-2, 0)$ and $(0, 3)$, and joining by a straight line, we obtain DE , the graph of $3x - 2y = -6$.

Since the two graphs meet in two points P and Q , there are two pairs of roots, which we find by measurement, $x = 1\frac{1}{3}, y = 4\frac{2}{3}$, or $x = -4, y = -3$.



Ex. 2. Solve graphically the following system:

$$\begin{cases} xy = 12, & (1) \\ x - y = 2. & (2) \end{cases}$$

From (1) $y = \frac{12}{x}$. Hence by substituting for x the values $-12, -11, \dots$ to $+12$, we obtain the following points: $(-12, -1), (-11, -1\frac{1}{11}), (-10, -1\frac{1}{5}), (-9, -1\frac{1}{3}), (-8, -1\frac{1}{2}), (-7, -1\frac{1}{7}), (-6, -2), (-5, -2\frac{2}{5}), (-4, -3), (-3, -4), (-2, -6), (-1, -12), (0, \pm \infty), (1, 12), (2, 6)$, etc., to $(12, 1)$.

Locating these points and joining them produces the graph of (1), which consists of two separate branches, CD and EF .

Locating two points of equation (2) and joining by a straight line, we have the graph AB of the equation (2).

The coördinates of the two points of intersection P and P' are the required roots. By actual measurement we find $x = 4.5+$, $y = 2.5+$, or $x = -2.5$, $y = -4.6$.

To obtain a greater degree of accuracy, the portion of the diagram near P is represented on a larger scale in the small diagram. Since the small part of CD which is represented is almost a straight line, it is sufficient to locate 2 or 3 points of this line. By actual measurement we find :

$$x = 4.606, y = 2.606.$$

Evidently the second pair is

$$x = -2.606, y = -4.606.$$

By increasing the scale further and further, any degree of accuracy may be obtained.

EXERCISE 116

Solve graphically the following simultaneous equations:

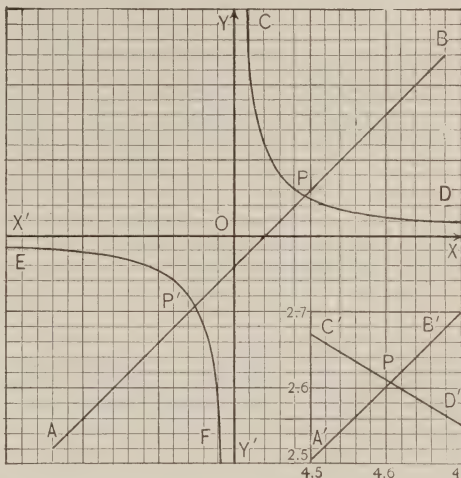
1.
$$\begin{cases} 3x + 4y = 8, \\ 2x - 3y = 6. \end{cases}$$

4.
$$\begin{cases} 4x + 3y = 12, \\ x + 5y = 6. \end{cases}$$

2.
$$\begin{cases} 3x + 4y = 10, \\ 4x + y = 9. \end{cases}$$

5.
$$\begin{cases} 3x + 5y = 7. \\ 5x - y = 7. \end{cases}$$

3.
$$\begin{cases} 2x - 3y = 7, \\ 3x + 2y = -8. \end{cases}$$



6. Show graphically that the following system cannot have finite roots:

$$\begin{cases} 2x - y = 2, \\ 2x - y = 4. \end{cases}$$

7. What are the relative positions of the graphs of two linear inconsistent equations?

8. Show graphically that the following system is satisfied by an infinite number of roots:

$$\begin{cases} 3x - 2y = 3, \\ 6x - 4y = 6. \end{cases}$$

Solve graphically:

$$9. \begin{cases} x^2 + y^2 = 16, \\ x + y = 2. \end{cases}$$

$$10. \begin{cases} x^2y = 12, \\ x - y = 1. \end{cases}$$

$$11. \begin{cases} x^2 + y^2 = 4, \\ x = 2y. \end{cases} \quad 12. \begin{cases} 2x - y + 1 = 0, \\ x + y + 1 = 0. \end{cases} \quad 13. \begin{cases} xy = 6, \\ x^2 + y^2 = 25. \end{cases}$$

TABLE OF SQUARES, CUBES, AND SQUARE ROOTS

x	x^2	x^3	\sqrt{x}	x	x^2	x^3	\sqrt{x}
1	1	1	1.000	21	4 41	9 261	4.583
2	4	8	1.414	22	4 84	10 648	4.690
3	9	27	1.732	23	5 29	12 167	4.796
4	16	64	2.000	24	5 76	13 824	4.899
5	25	125	2.236	25	6 25	15 625	5.000
6	36	216	2.449	26	6 76	17 576	5.099
7	49	343	2.646	27	7 29	19 683	5.196
8	64	512	2.828	28	7 84	21 952	5.292
9	81	729	3.000	29	8 41	24 389	5.385
10	1 00	1 000	3.162	30	9 00	27 000	5.477
11	1 21	1 331	3.317	31	9 61	29 791	5.568
12	1 44	1 728	3.464	32	10 24	32 768	5.657
13	1 69	2 197	3.606	33	10 89	35 937	5.745
14	1 96	2 744	3.742	34	11 56	39 304	5.831
15	2 25	3 375	3.873	35	12 25	42 875	5.916
16	2 56	4 096	4.000	36	12 96	46 656	6.000
17	2 89	4 913	4.123	37	13 69	50 653	6.083
18	3 24	5 832	4.243	38	14 44	54 872	6.164
19	3 61	6 859	4.359	39	15 21	59 319	6.245
20	4 00	8 000	4.472	40	16 00	64 000	6.325

CHAPTER XVIII

QUADRATIC EQUATIONS INVOLVING ONE UNKNOWN QUANTITY

315. A quadratic equation, or equation of the second degree, is an integral rational equation that contains the square of the unknown number, but no higher power; *e.g.* $x^2 - 4x = 7$, $6y^2 = 17$, $ax^2 + bx + c = 0$.

316. A complete, or affected, quadratic equation is one which contains both the square and the first power of the unknown quantity.

317. A pure, or incomplete, quadratic equation contains only the square of the unknown quantity.

$ax^2 + bx + c = 0$ is a complete quadratic equation.

$ax^2 = m$ is a pure quadratic equation.

PURE QUADRATIC EQUATIONS

318. A pure quadratic is solved by reducing it to the form $x^2 = a$, and extracting the square root of both members.

Ex. 1. Solve $13x^2 - 19 = 7x^2 + 5$.

Transposing, etc., $6x^2 = 24$.

Dividing, $x^2 = 4$.

Extracting the square root of each member,

$$x = +2 \text{ or } x = -2.$$

This answer is frequently written $x = \pm 2$.

Check. $13(\pm 2)^2 - 19 = 33$; $7(\pm 2)^2 + 5 = 33$.

Ex. 2. Solve $\frac{a-x}{a+x} = \frac{x+4a}{x-4a}$.

Clearing of fractions, $ax - x^2 - 4a^2 + 4ax = ax + 4a^2 + x^2 + 4ax$.

Transposing and combining, $-2x^2 = 8a^2$.

Dividing by -2 , $x^2 = -4a^2$.

Extracting the square root, $x = \pm \sqrt{-4a^2}$,

or $x = \pm 2a\sqrt{-1}$.

EXERCISE 117

Solve the following equations:

1. $17x^2 - 7 = 418$.

9. $\frac{x}{x+1} + \frac{x}{x-1} = 2\frac{2}{3}$.

2. $13x^2 - 19 = 7x^2 + 5$.

10. $\frac{x+3}{x-3} + \frac{x-3}{x+3} = 3\frac{1}{3}$.

3. $\frac{15x}{2} = \frac{810}{3x}$.

11. $\frac{4x+5}{7x-1} = \frac{x+2}{5x-3}$.

4. $ax^2 - b = c$.

5. $(x + \frac{1}{2})(x - \frac{1}{2}) = \frac{5}{16}$.

12. $\sqrt{23x^2 - 70x + 81} = 5x - 7$

6. $x^2 + 7 = 4$.

13. $3 - \sqrt{7x^2 - 24x + 45} = 4x$.

7. $4x(x-1) = -4(x-2)$.

14. $\frac{1}{\sqrt{x+4}} = \frac{\sqrt{x-4}}{3}$.

8. $\frac{x+2}{x-2} + \frac{x-2}{x+2} = \frac{24}{x^2-4}$.

15. $\sqrt[3]{\sqrt{5x^2+9}-19} = 2$.

16. $2 + \sqrt{17-2x} = \sqrt{2x+17}$.

17. $(a+x)(b-x) + (a-x)(b+x) = 0$.

18. $(a+bx)^2 + (ax-b)^2 = 2(a^2x^2 - b^2)$.

19. $\frac{ax+b}{a+bx} = \frac{cx+d}{c+dx}$.

20. If $a^2 + b^2 = c^2$, find a in terms of b and c .

21. If $d = \frac{g}{2}t^2$, solve for t .

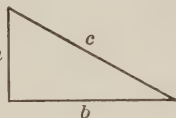
22. If $2a^2 + 2b^2 = 4m^2 + c^2$, solve for m .

23. Solve the equation of the preceding example for c .
 24. If $a^2 = b^2 + c^2 - 2cp$, solve for b .
 25. If $S = \pi r^2$, solve for r .
 26. If $S = 4\pi r^2$, solve for r .
 27. If $l:g = t^2:\pi^2$, solve for t .
 28. If $R = \frac{Kl}{d^2}$, solve for d .
 29. If $E = \frac{mv^2}{2}$, solve for v .

EXERCISE 118

1. Find a positive number which is equal to its reciprocal.
2. The ratio of two numbers is 5:4, and their product is 980. Find the numbers.
3. Three numbers are to each other as 2:3:4, and the sum of their squares is 261. Find the numbers.
4. Two numbers are as 5:4, and the difference of their squares is 36. Find the numbers.
5. The sides of two square fields are as 7:24, and they contain together 10,000 square yards. Find the side of each field.

319. A *right triangle* is a triangle, one of whose angles is a right one. The side opposite the right angle is called the *hypotenuse* (c in the diagram). If the hypotenuse contains c units of length, and the two other sides respectively a and b units, then



$$c^2 = a^2 + b^2.$$

Since such a triangle may be considered one half of a rectangle, its area contains $\frac{ab}{2}$ square units.

6. The hypotenuse of a right triangle is 15 inches, and the two other sides are as 3:4. Find the sides.
7. The hypotenuse of a right triangle is to one side as 41:9, and the third side is 40 centimeters. Find the unknown sides and the area.

8. The hypotenuse of a right triangle is 2, and the other two sides are equal. Find these sides.

9. The area of a rectangle is 1260 square feet, and the two sides are as 5:7. What is the length of the sides?

10. The area of a right triangle is 147 square feet, and the two smaller sides are as 2:3. Find the sides.

11. The area of a circle exceeds the area of another circle by 616 square inches, and their radii are as 25:24. If the value of π is assumed equal to $\frac{22}{7}$, find the radii of the circles. (The area S of a circle whose radius is R is determined by the formula $S = \pi R^2$.)

12. Two circles together contain 44,506 square inches, and their radii are as 15:8. Find the radii.

COMPLETE QUADRATIC EQUATIONS

320. Method of completing the square. The following example illustrates the method of solving a complete quadratic equation by completing the square.

$$\text{Solve } x^2 - 7x + 10 = 0.$$

$$\text{Transposing, } x^2 - 7x = -10.$$

The left member can be made a complete square by adding another term. To find this term, let us compare $x^2 - 7x$ with the perfect square $x^2 - 2mx + m^2$. Evidently 7 takes the place of $2m$, or $m = \frac{7}{2}$. Hence to make $x^2 - 7x$ a complete square we have to add $(\frac{7}{2})^2$ which corresponds with m^2 .

Adding $(\frac{7}{2})^2$ to each member,

$$x^2 - 7x + (\frac{7}{2})^2 = \frac{49}{4} - 10,$$

or

$$(x - \frac{7}{2})^2 = \frac{9}{4}.$$

Extracting square roots, $x - \frac{7}{2} = \pm \frac{3}{2}$.

$$\text{Hence } x = \frac{7}{2} \pm \frac{3}{2}.$$

$$\text{Therefore } x = 5 \text{ or } x = 2.$$

$$\text{Check. } 5^2 - 7 \cdot 5 + 10 = 0, \quad 2^2 - 7 \cdot 2 + 10 = 0.$$

Ex. 1. Solve $9x^2 - 2 = 15x + 4$.

Transposing, $9x^2 - 15x = 6$.

Dividing by 9, $x^2 - \frac{5}{3}x = \frac{2}{3}$.

Completing the square (*i.e.* adding $(\frac{5}{6})^2$ to each member),

$$x^2 - \frac{5}{3}x + (\frac{5}{6})^2 = \frac{2}{3} + \frac{25}{36}.$$

Simplifying, $(x - \frac{5}{6})^2 = \frac{49}{36}$.

Extracting square roots, $x - \frac{5}{6} = \pm \frac{7}{6}$.

Transposing, $x = \frac{5}{6} \pm \frac{7}{6}$.

Therefore $x = 2$, or $-\frac{1}{3}$.

321. Hence to solve a complete quadratic:

Reduce the equation to the form $x^2 + px = q$. Complete the square by adding the square of one half the coefficient of x . Extract the square root and solve the equation of the first degree thus formed.

Ex. 2. Solve $\frac{x-1}{a} + \frac{2a+1}{x} = 2$.

Clearing of fractions, $x^2 - x + 2a^2 + a = 2ax$.

Transposing, $x^2 - x - 2ax = -2a^2 - a$.

Uniting, $x^2 - x(1 + 2a) = -2a^2 - a$.

Completing the square,

$$x^2 - x(1 + 2a) + \left(\frac{1 + 2a}{2}\right)^2 = -2a^2 - a + \frac{1 + 4a + 4a^2}{4}.$$

Simplifying, $\left(x - \frac{1 + 2a}{2}\right)^2 = \frac{1 - 4a^2}{4}$.

Extracting square root, $x - \frac{1 + 2a}{2} = \pm \frac{1}{2}\sqrt{1 - 4a^2}$.

Transposing, $x = \frac{1 + 2a}{2} \pm \frac{1}{2}\sqrt{1 - 4a^2}$.

Therefore $x = \frac{1 + 2a \pm \sqrt{1 - 4a^2}}{2}$.

Ex. 3. Solve $a^2x^2 - x^2 + 2a^2x + a^2 = 0$.

Transposing, $a^2x^2 - x^2 + 2a^2x = -a^2$.

Uniting, $x^2(a^2 - 1) + 2a^2x = -a^2$.

Dividing, $x^2 + \frac{2a^2}{a^2 - 1}x = -\frac{a^2}{a^2 - 1}$.

Completing the square,

$$x^2 + \frac{2a^2}{a^2 - 1}x + \left(\frac{a^2}{a^2 - 1}\right)^2 = \frac{a^4}{(a^2 - 1)^2} - \frac{a^2}{(a^2 - 1)}.$$

Simplifying, $\left(x + \frac{a^2}{a^2 - 1}\right)^2 = \frac{a^2}{(a^2 - 1)^2}$.

Extracting the square roots,

$$x + \frac{a^2}{a^2 - 1} = \pm \frac{a}{a^2 - 1}.$$

Transposing, $x = -\frac{a^2}{a^2 - 1} \pm \frac{a}{a^2 - 1}$.

Therefore

$$x = -\frac{a^2 + a}{a^2 - 1} = -\frac{a}{a - 1} \text{ or } x = \frac{-a^2 + a}{a^2 - 1} = -\frac{a}{a + 1}.$$

EXERCISE 119

- | | |
|--------------------------|---|
| 1. $x^2 + 8x = 33$. | 13. $10x^2 = 27 - 39x$. |
| 2. $x^2 - 6x = 7$. | 14. $28x^2 - 65 = 17x$. |
| 3. $x^2 - 6x - 16 = 0$. | 15. $27x^2 - 30x = 77$. |
| 4. $x^2 + 12x = 64$. | 16. $35x^2 - 65 = 66x$. |
| 5. $x^2 + 11 = 12x$. | 17. $8x^2 + 41x + 36 = 0$. |
| 6. $x^2 - 21 = 4x$. | 18. $12x^2 + 77x + 30 = 0$. |
| 7. $x^2 - 11x = 60$. | 19. $x(x + 5) - 84 = 0$. |
| 8. $x^2 - 8x + 15 = 0$. | 20. $3x^2 - 3x - 330 = 0$. |
| 9. $x^2 + 15 = 12x$. | 21. $2x^2 - 5x - 3 = 0$. |
| 10. $x^2 + 17x = -70$. | 22. $6x^2 + 1 = 5x$. |
| 11. $12x^2 + x = 35$. | 23. $7x + \frac{1}{4} = \frac{4x + 7}{16x}$. |
| 12. $15x^2 = 7x + 88$. | |

$$24. \quad 9x^2 = x + \frac{2}{3}.$$

$$25. \quad x + \frac{6}{x} = 5.$$

$$26. \quad x - \frac{18}{x} = 3.$$

$$27. \quad 3x - \frac{16}{x} = 8.$$

$$28. \quad 5x - \frac{22}{2x+3} = 18.$$

$$35. \quad (2x+1)^2 + (3x+1)^2 = (2x+3)^2.$$

$$36. \quad 3(x^2-1) - 24 = 4(x+5)(x-3).$$

$$29. \quad 3x + \frac{27}{5x-1} = 9.$$

$$30. \quad 11x - \frac{42}{3x+1} = 16.$$

$$31. \quad 3x^2 - \frac{5}{4}x + \frac{1}{8} = 0.$$

$$32. \quad (3x-5)(7x-8) = 140.$$

$$33. \quad (2x-15)(3x+8) = -154.$$

$$34. \quad (x+3)^2 + (x+5)^2 = 514.$$

$$37. \quad \frac{x}{2(x-3)} = \frac{x-3}{x-1}.$$

$$46. \quad \frac{8}{x-4} + \frac{5}{x-3} = 3.$$

$$38. \quad \frac{7x-5}{10x-3} = \frac{5x-3}{6x+1}.$$

$$47. \quad \frac{9}{x-2} - \frac{4}{x-7} = 5.$$

$$39. \quad \frac{x-8}{x+2} = \frac{x-1}{2x+10}.$$

$$48. \quad \frac{5}{x-2} - \frac{8}{x-6} = 9.$$

$$40. \quad \frac{5x-1}{9} + \frac{3x-1}{5} = \frac{2}{x} + x - 1.$$

$$49. \quad \frac{x-3}{x-4} + \frac{x+1}{x-2} = 2\frac{3}{4}.$$

$$41. \quad x^2 + 22(x+5) = 0.$$

$$50. \quad \frac{2x+3}{x-3} - \frac{x+2}{x-2} = 3.$$

$$42. \quad \frac{3}{4}x^2 - x = \frac{1}{2}\frac{9}{4}.$$

$$43. \quad \frac{2x(2x-5)}{2x-1} - \frac{2}{2x-1} = 3.$$

$$51. \quad \frac{5x-6}{2x-3} - \frac{7x+2}{5x-4} + 8 = 0$$

$$44. \quad \frac{x-5}{x+1} = x.$$

$$52. \quad x^2 + 2ax = 3a^2.$$

$$53. \quad x^2 + 11m^2 = 12mx.$$

$$45. \quad \frac{x-1}{x-13} = \frac{1}{x}.$$

$$54. \quad 3x^2 - 2n^2 = 5xn.$$

$$55. \quad 2x^2 - 5ax = 3a^2.$$

$$56. \quad x^2 + (a+b)x = 2a^2 - 5ab + 2b^2.$$

57. $x^2 + ab = ax + bx.$

60. $a + x = \frac{1}{a} + \frac{1}{x}.$

58. $x + \frac{1}{x} = a + \frac{1}{a}.$

61. $\frac{a+x}{b+x} + \frac{b+x}{a+x} = \frac{5}{2}.$

59. $x^2 - (a-x)^2 = (a-2x)^2.$

62. $ax^2 + bx + c = 0.$

322. Solution by formula. Every quadratic equation can be reduced to the general form,

$$ax^2 + bx + c = 0.$$

Solving this equation by the method of the preceding article, we obtain

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The roots of any quadratic equation may be obtained by substituting the values of a , b , and c , in the general answer.

Ex. 1. Solve $5x^2 = 26x - 5.$

Transposing, $5x^2 - 26x + 5 = 0.$

Hence $a = 5$, $b = -26$, $c = 5.$

Therefore
$$x = \frac{+26 \pm \sqrt{(-26)^2 - 4 \cdot 5 \cdot 5}}{10}$$

$$= \frac{26 \pm 24}{10} = 5 \text{ or } \frac{1}{5}.$$

Ex. 2. Solve $px^2 - p^2x - x = -p.$

Reducing to general form, $px^2 - (p^2 + 1)x + p = 0.$

Hence $a = p$, $b = -(p^2 + 1)$, $c = p.$

Therefore
$$x = \frac{p^2 + 1 \pm \sqrt{(p^2 + 1)^2 - 4p^2}}{2p}$$

$$= \frac{p^2 + 1 \pm (p^2 - 1)}{2p} = p \text{ or } \frac{1}{p}.$$

EXERCISE 120

Solve by the above formula:

1. $3x^2 - 10x + 3 = 0.$
2. $7x^2 + 9x - 10 = 0.$
3. $x^2 - 5x + 3 = 0.$
4. $2x^2 - 7x + 3 = 0.$
5. $3x^2 - 2 = 5x.$
6. $15x^2 - 53x = 42.$
7. $8x^2 - 17x = 115.$
8. $9x^2 - 32x - 16 = 0.$
9. $7x^2 - 39 = 8x.$
10. $6x^2 = 37x - 6.$
11. $6x^2 = x + 12.$
12. $20 = x^2 - x.$
13. $10 + 16x = -6x^2.$
14. $12x^2 + x = 6.$
15. $15x^2 + 14x = -3.$
16. $5x - x^2 = -50.$
17. $3x^2 - 5x = 2.$
18. $9x^2 + 17x = 310.$
19. $x^2 + \frac{2}{3}x = 40.$
20. $x^2 - \frac{5}{6}x = -\frac{1}{6}.$
21. $4x^2 - 12qx + 9q^2 = 0.$
22. $6x^2 - 49mx + 8m^2 = 0.$
23. $6x^2 + 5nx = 6n^2.$
24. $4x^2 - 6dx = 4d^2.$
25. $x^2 + 2ax = b.$
26. $x^2 - 2ax + b = 0.$
27. $abx^2 - (a^2 + b^2)x + ab = 0.$
28. $x^2 - 4ax + 4a^2 - b^2 = 0.$
29. $x^2 - 2ax = 4b^2 - a^2.$
30. $a^2x^2 - b^2x^2 - 4abx = a^2 - b^2.$
31. $m^2x^2 - m(a - b)x - ab = 0.$
32. $(n - p)x^2 + (p - m)x + (m - n) = 0.$

Find the roots of the following equations to two decimal places:

33. $x^2 + x - 1 = 0.$
34. $2x^2 + 2x - 1 = 0.$
35. $x^2 - 4x + 2 = 0.$
36. $x^2 - 3x + 1 = 0.$
37. $6x^2 - 5x + 1 = 0.$
38. $.2x^2 - .5x = .3.$
39. $\frac{5}{x} + \frac{8}{x+1} = 9.$
40. $9x^2 = x + \frac{2}{3}.$

323. Solution by factoring. Let it be required to solve the equation :

$$5x^2 + 5 = 26x;$$

or, transposing all terms to one member,

$$5x^2 - 26x + 5 = 0.$$

Resolving into factors,

$$(5x - 1)(x - 5) = 0.$$

Now, if either of the factors $5x - 1$, $x - 5$ is zero, the product is zero. Therefore the equation will be satisfied if x has such a value that either

$$5x - 1 = 0, \quad (1)$$

or

$$x - 5 = 0. \quad (2)$$

Solving (1) and (2) we obtain the roots

$$x = \frac{1}{5} \text{ or } x = 5.$$

324. Evidently this method can be applied to equations of any degree, if one member of the equation is zero and the other member can be factored.

Ex. 1. Solve $x^3 = \frac{7x^2 + 15x}{2}$.

Clearing of fractions, $2x^3 = 7x^2 + 15x$.

Transposing, $2x^3 - 7x^2 - 15x = 0$.

Factoring, $x(2x + 3)(x - 5) = 0$.

Therefore $x = 0$, $2x + 3 = 0$, or $x - 5 = 0$.

Hence the equation has three roots, 0, $-\frac{3}{2}$, and 5.

Ex. 2. Solve $x^3 - 3x^2 - 4x + 12 = 0$.

Factoring, $x^2(x - 3) - 4(x - 3) = 0$.

$$(x^2 - 4)(x - 3) = 0.$$

Or $(x - 2)(x + 2)(x - 3) = 0$.

Hence the roots are 2, -2, 3.

Ex. 3. Solve $x^3 + 2x^2 - 8x + 5 = 0$.

By the factor theorem we find that $x = 1$ satisfies the equation, and is, therefore, a root of the equation.

Dividing by $x - 1$ we obtain the other factor, and have

$$(x - 1)(x^2 + 3x - 5) = 0.$$

Hence $x - 1 = 0$, or $x^2 + 3x - 5 = 0$.

Solving the second equation by the formula,

$$x = \frac{-3 \pm \sqrt{29}}{2}, \text{ or } x = 1.$$

Ex. 4. Solve $x^3 + a^3 = 0$.

Factoring, $(x + a)(x^2 - ax + a^2) = 0$.

Hence $x + a = 0$, or $x^2 - ax + a^2 = 0$.

Therefore $x = -a$, or $\frac{a \pm a\sqrt{-3}}{2}$.

Ex. 5. Form an equation whose roots are 4 and -6 .

The equation is evidently $(x - 4)(x - (-6)) = 0$.

I.e. $(x - 4)(x + 6) = 0$.

Or $x^2 + 2x - 24 = 0$.

325. If both members of an equation are divided by an expression involving the unknown quantity, the resulting equation contains fewer roots than the original one. In order to obtain all roots of the original equation, such a common divisor must be made equal to zero, and the equation thus formed be solved. *E.g.* let it be required to solve

$$x^2 - 9 = 5(x - 3).$$

If we divide both members by $x - 3$, we obtain $x + 3 = 5$ or $x = 2$. But evidently the value $x = 3$ obtained from the equation $x - 3 = 0$ is also a root, for $x^2 - 9 - 5(x - 3) = 0$, or $(x - 3)(x + 3 - 5) = 0$. Therefore $x = 3$ or $x = 2$.

EXERCISE 121

Solve by factoring:

1. $x^2 - 6x = 7$

12. $x^3 = 5x^2 + 4x$.

2. $x^2 + 8x = 20$.

13. $(x-1)(x^2 - 12x + 27) = 0$.

3. $x^2 - 16 = 6x$.

14. $x(x-6) = 7x - 42$.

4. $x(x+5) = 14$.

15. $x^3 + x = 5x(2x-4)$.

5. $x^2 - 7x = 8$.

16. $x^4 - 16 = 0$.

6. $x^2 - 9x + 20 = 0$.

17. $x^3 + x^2 - 4x - 4 = 0$.

7. $x^2 + 29x = 210$.

18. $5x^3 - 2x^2 - 10x + 4 = 0$.

8. $x^3 - 11x^2 + 18x = 0$.

19. $3x^3 + 13x^2 = 30x$.

9. $4x^3 + 17x^2 + 4x = 0$.

20. $3x^2 + 7x = 6$.

10. $x^3 - 8 = 0$.

21. $6x^2 + 7x = 3$.

11. $x^2(10x+1) = 3x$.

22. $6x(x-3) = 5(x-3)$.

23. $(a-x)(x-b) = (a-x)(c-x)$.

24. $a^2 - x^2 = (a-x)(b+c-x)$.

25. $x^2 = (a+b)x$.

27. $x(x^2-1) = 5(x-1)$.

26. $2x^3 + x^2 - 5x + 2 = 0$.

28. $x^3 - 6x^2 + 11x - 6 = 0$.

29. $x^3 - 4x^2 - 20x + 48 = 0$.

30. Find all roots of the equation $x^3 = 1$.

31. Find all values of $\sqrt[3]{8}$.

(HINT. Let $x = \sqrt[3]{8}$, hence $x^3 = 8$, etc.)

Solve:

32. $\frac{7}{x+5} - \frac{8}{x-6} = \frac{3}{x-1}$.

34. $\frac{3}{x-2} + \frac{4}{x-3} = \frac{6}{7-x}$.

33. $\frac{5}{x-4} + \frac{6}{x-3} = \frac{6}{x-6}$.

35. $\frac{7}{x-3} + \frac{9}{x+5} = \frac{40}{x+1}$.

$$36. 3x - 8 = 4\sqrt{4x + 1}.*$$

$$39. 6 + 5\sqrt{13x + 9} = 8x.$$

$$37. 4x - 3\sqrt{7x - 6} = 6.$$

$$40. \sqrt{x + 5} + \sqrt{x - 3} = 4.$$

$$38. 5 + 3\sqrt{3x - 5} = 2x.$$

$$41. \sqrt{x - 3} + \sqrt{x - 8} = 5.$$

$$42. \sqrt{x + 7} - \sqrt{x - 5} = 2.$$

$$43. \sqrt{x + 2} + \sqrt{x - 3} = \sqrt{3x + 4}.$$

$$44. \sqrt{x + 7} + \sqrt{2x + 7} = \sqrt{8x + 9}.$$

$$45. \sqrt{3x + 4} - \sqrt{5x - 11} = \sqrt{x - 3}$$

$$46. \frac{x+a}{x-a} + \frac{x-a}{x+a} = 3\frac{1}{3}.$$

$$47. \frac{3b}{x-a} - \frac{2a}{x-b} = \frac{2a+b}{x+a}.$$

$$48. \frac{x^3 - 10x^2 + 1}{x^2 - 6x + 9} = x - 3.$$

$$49. \frac{x-a}{a-b} + \frac{x+b}{2a} = \frac{b}{2a-x} + \frac{x-b}{2a-2b}.$$

$$50. x - b\sqrt{x} = a(a-b).$$

$$53. \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = \frac{a}{x}.$$

$$51. \frac{ax+b}{bx+a} = \frac{mx-n}{nx-m}.$$

$$52. cx + \frac{ac}{a+b} = (a+b)x^2.$$

$$54. \sqrt{\frac{a-x}{b+x}} + \sqrt{\frac{b+x}{a-x}} = c + \frac{1}{c}.$$

Form the equations whose roots are:

$$55. 2, 1.$$

$$57. -4, -5.$$

$$59. 1, 2, 3.$$

$$56. 2, -3.$$

$$58. 0, 3.$$

$$60. -1, 2, 0.$$

PROBLEMS INVOLVING QUADRATICS

326. Problems involving quadratics have in general two answers, but frequently the conditions of the problem exclude negative or fractional answers, and consequently many problems of this type have only one solution.

* The answers of radical equations must be substituted in the original equation in order to determine which roots are true roots. (See § 285.)

EXERCISE 122

1. The square of a number increased by 14 times the number equals 275. Find the number.
2. What number increased by its reciprocal equals $\frac{37}{6}$?
3. The difference of two numbers is 49, and their reciprocal values differ by $\frac{1}{8}$. Find the numbers.
4. What number exceeds its reciprocal by n ? Find the number if (a) $n = \frac{5}{6}$, (b) $n = \frac{3}{2}$.
5. Find two numbers whose difference is 7, and whose product is 30.
6. Find two factors of 1728 whose sum is 96.
7. Find two consecutive numbers, the sum of whose squares equals 113.
8. A man sold a horse for \$96, and gained as many per cent as the horse cost dollars. Find the cost of the horse.
9. A man sold a watch for \$16, and lost as many per cent as the watch cost dollars. Find the cost of the watch.
10. A needs 3 days more than B to do a certain piece of work, and, working together, the two men can do it in 2 days. In how many days can B do the work?
11. A cistern is filled by two pipes in 40 minutes, and the larger pipe can fill it in one hour less time than the smaller one. In how many minutes can the smaller pipe fill the cistern?
12. A number of boys bought a boat, each paying as many dollars as there were boys. Had there been 8 boys more, each would have paid \$4 less. How many dollars did each boy pay?
13. If a train had traveled 5 miles an hour faster, it would have needed one hour less to travel 150 miles. Find the rate of the train.

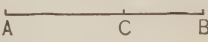
14. A wheelman, traveling a distance of 72 kilometers, would arrive 30 minutes earlier if he traveled 2 kilometers per hour faster. Find his rate of traveling.

15. To pave a room with square tiles of a certain size, 360 tiles are needed. If each tile were one inch longer and one inch wider, 250 tiles would be needed. Find the dimensions of the tiles.

16. A rectangular field has an area of 180,000 square feet, and a perimeter of 1720 feet. Find the dimensions of the field.

17. Find the price of an egg, if 1 more for 6¢ would diminish the price of 100 eggs by \$1.

18. One side of a rectangular field exceeds the other side by 70 feet, and the area equals 49,400 square feet. Find the dimensions.

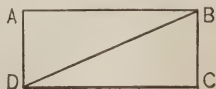
19. On the prolongation of a line AC , 10 inches long, a point B is taken, so that the rectangle, constructed with AB and CB as sides, contains  39 square inches. Find AB and CB .

20. A line AB , 20 inches long, is divided into two parts, AC and CB , so that AC is the mean proportional between CB and AB . Find the length of AC . (AB is said to be divided in "extreme and mean ratio.")

21. Solve the preceding problem if AB equals a inches.

22. A line AB is divided into two parts, AC and CB , so that AC is the mean proportional between CB and AB . If AC equals 1 inch, find AB .

23. The length AB of a rectangle, $ABCD$, exceeds its width AD by 119 feet, and the line BD joining two opposite vertices (called "diagonal") equals 221 feet. Find AB and AD .



24. A rectangular grass plot, 40 feet long and 30 feet wide, is surrounded by a walk of uniform width. If the area of the walk is $\frac{2}{3}$ of the area of the plot, how wide is the walk?

25. A circular basin is surrounded by a path 6 feet wide, and the area of the path is $\frac{7}{9}$ of the area of the basin. Find the radius of the basin.

26. A number is formed by two digits, and the tens digit exceeds the units digit by 2. If the number is divided by the product of its digits, the quotient is $2\frac{2}{3}$. Find the number.

27. A man bought a certain number of pounds of tea and 10 pounds more of coffee, paying \$14.00 for the tea and \$9.00 for the coffee. If a pound of tea cost 40¢ more than a pound of coffee, what was the price of the coffee per pound?

28. A stone is dropped into a well, and the sound of its impact upon the water is heard at the top of the well $4\frac{2}{9}$ seconds later. If the velocity of sound is assumed as 360 meters per second, and $g = 10$ meters, how deep is the well? (A body falls in t seconds $\frac{g}{2}t^2$ meters.)

EQUATIONS IN THE QUADRATIC FORM

327. An equation is said to be in the **quadratic form** if it contains only two unknown terms, and the unknown factor of one of these terms is the square of the unknown factor of the other, as

$$ax^{2n} + bx^n + c = 0, \quad x^5 - 3x^{\frac{5}{2}} = 7, \quad (x^2 - 1)^2 - 4(x^2 - 1) = 9.$$

328. Equations in the quadratic form can be solved by the methods used for quadratics.

Ex. 1. Solve $x^6 - 9x^3 + 8 = 0.$

By formula,

$$x^3 = \frac{9 \pm \sqrt{81 - 32}}{2}$$

$$= \frac{9 \pm 7}{2} = 8, \text{ or } 1.$$

Therefore $x = \sqrt[3]{8} = 2$, or $x = \sqrt[3]{1} = 1$.

If the factoring method is used, we have

$$(x^3 - 8)(x^3 - 1) = 0.$$

Writing each factor equal to zero, and solving,

$$x = 2, x = -1 \pm \sqrt{-3}, x = 1, x = \frac{-1 \pm \sqrt{-3}}{2}.$$

NOTE. The solution by means of factoring frequently produces all roots, while other methods produce only some roots of the equation.

Ex. 2. Solve $x^{-\frac{5}{2}} - 33x^{-\frac{5}{4}} + 32 = 0$.

Factoring, $(x^{-\frac{5}{4}} - 32)(x^{-\frac{5}{4}} - 1) = 0$.

Therefore $x^{-\frac{5}{4}} = 32$, or 1.

Raising both members to the $-\frac{4}{5}$ power,

$$x = 32^{-\frac{4}{5}} \text{ or } 1^{-\frac{4}{5}} = \frac{1}{16} \text{ or } 1.$$

EXERCISE 123

Solve the following equations:

1. $x^4 - 5x^2 = 36$.
2. $12x^2 - 32 = x^4$.
3. $x^6 - 26x^3 - 27 = 0$.
4. $x^6 - 60x^3 = 256$.
5. $50x^4 + 23x^2 - 1 = 0$.
6. $(x^2 - 10)(x^2 - 3) = 78$.
7. $(x^2 - 5)^2 + (x^2 - 1)^2 = 40$.
8. $x^{-4} + 16 = 17x^{-2}$.
9. $8\sqrt{x} = 15x^{-\frac{1}{2}} - 14$.
10. $3x^{\frac{1}{6}} + 3x^{-\frac{1}{6}} = 10$.
11. $x - 2\sqrt{x} - 3 = 0$.
12. $x + 3\sqrt{x} = 10$.
13. $x - 5\sqrt{x} + 6 = 0$.
14. $x^{\frac{2}{3}} + 3x^{\frac{1}{3}} = 10$.
15. $\sqrt[3]{x^2} - 2\sqrt[3]{x} = 24$.
16. $3\sqrt[5]{x^2} + 2\sqrt[5]{x} - 16 = 0$.
17. $x^{2n} - 3x^n + 2 = 0$.
18. $x^4 - 2ax^2 + a^2 - b^2 = 0$.
19. $x^{\frac{1}{2}} + 6x^{-\frac{1}{2}} = 5$.
20. $3x^{\frac{2n}{3}} + 2 = 16x^{-\frac{2n}{3}}$.
21. $\sqrt{x-1} = x-1$.
22. $x - (a+b)\sqrt{x} = 2a(a-b)$.
23. $\sqrt[4]{x} + \sqrt{x} = 20$.
24. $x + ab = (a+b)\sqrt{x} + 2(a-b)^2$.

329. In more complex examples it is advantageous to substitute a letter for an expression involving x .

Ex. 1. Solve $\left(\frac{x+4}{x}\right)^2 + 15 = \frac{8(x+4)}{x}$.

Let $\frac{x+4}{x} = y$.

Then $y^2 + 15 = 8y$,

or $y^2 - 8y + 15 = 0$,

or $(y-5)(y-3) = 0$.

Hence $y = 5$, or $y = 3$.

I.e. $\frac{x+4}{x} = 5$, or $\frac{x+4}{x} = 3$.

Solving, $x+4 = 5x$, $x+4 = 3x$,
 $4x = 4$, $2x = 4$,
 $x = 1$, $x = 2$.

Ex. 2. Solve $x^2 - 8x - 2\sqrt{x^2 - 8x + 40} = -5$.

Adding 40, $x^2 - 8x + 40 - 2\sqrt{x^2 - 8x + 40} = 35$.

Let $\sqrt{x^2 - 8x + 40} = y$, then $x^2 - 8x + 40 = y^2$.

Hence $y^2 - 2y = 35$.

$y^2 - 2y - 35 = 0$.

$(y-7)(y+5) = 0$.

Therefore $y = 7$, or $y = -5$.

$\sqrt{x^2 - 8x + 40} = 7$, or $\sqrt{x^2 - 8x + 40} = -5$,
 $x^2 - 8x + 40 = 49$, $x^2 - 8x + 40 = 25$,
 $x^2 - 8x - 9 = 0$, $x^2 - 8x + 15 = 0$,
 $(x-9)(x+1) = 0$, $(x-5)(x-3) = 0$,
 $x = 9$, or -1 , $x = 5$ or 3 .

Since both members of the equation were squared, some of the roots may be extraneous. Substituting, it will be found that 9 and -1 satisfy the equation, while 5 and 3 are extraneous roots.

This can be seen without substituting, for 5 and 3 are the roots of the equation $\sqrt{x^2 - 8x + 40} = -5$. But as the square root is restricted to its positive values, it cannot be equal to a negative quantity.

EXERCISE 124

1. $3(x^2 - 2)^2 + 6(x^2 - 2) = 24.$
2. $\left(x + \frac{1}{x}\right)^2 + 3\left(x + \frac{1}{x}\right) = 10.$
3. $x^2 + 5x - 5 = \frac{6}{x^2 + 5x}.$
4. $\left(x + \frac{2}{x}\right)^2 + 3x + \frac{6}{x} = 4.$
5. $\frac{x^2 + 1}{x^2 - 1} + \frac{x^2 - 1}{x^2 + 1} = \frac{34}{15}.$
6. $\frac{x^2 + 2}{x^2 - 2} + \frac{3(x^2 - 2)}{x^2 + 2} = 4.$
7. $x^2 + 2x - 5 = \frac{24}{x^2 + 2x}.$
8. $x^2 - 6x + 12 = \frac{-28}{x^2 - 6x + 1}.$
9. $x^2 - 4x - 3 = \frac{12}{x^2 - 4x + 1}.$
10. $(x^2 - 5x + 2)^2 + 6(x^2 - 5x + 2) + 8 = 0.$
11. $(x^2 + 3x - 2)^2 - 4(x^2 + 3x - 2) = 32.$
12. $2(x^2 + 2x + 3)^2 - (x^2 + 2x + 4)^2 = 98.$
13. $(x^2 + 2x - 1)(x^2 + 2x - 2) = 2.$
14. $(x^2 - 3)^2 + 4(x^2 - 3) = 5.$
15. $\frac{x^2 + m^2}{x^2 - m^2} + \frac{x^2 - m^2}{x^2 + m^2} = \frac{41}{20}.$
16. $\frac{x^2 + 6}{x} + \frac{35x}{x^2 + 6} = 12.$
17. $(3x^2 - 2)^2 - 5(3x^2 - 2) + 4 = 0.$
18. $\frac{\sqrt{x}}{21 - \sqrt{x}} + \frac{21 - \sqrt{x}}{\sqrt{x}} = 2\frac{1}{2}.$
19. $(x + \sqrt{x})^4 - (x + \sqrt{x})^2 = 20,592.$
20. $\sqrt{\frac{14+x}{14-x}} + \sqrt{\frac{14-x}{14+x}} = 3\frac{1}{3}.$
21. $\frac{x+3}{x-3} + 6 = 5\sqrt{\frac{x+3}{x-3}}.$
22. $\sqrt{x^2 + 3} + \frac{4}{\sqrt{x^2 + 3}} = 4.$
23. $x^2 - 6 = 2\sqrt{x^2 + 9}.$
24. $x^2 = \sqrt{x^2 - 7} + 13.$
25. $x^2 - 7 - 5\sqrt{x^2 - 7} = 24.$
26. $(x+3) + (x+3)^{\frac{1}{2}} = 2.$
27. $x - 3 - (x-3)^{\frac{1}{2}} = 6.$
28. $4x - 7 - 2\sqrt{4x - 7} = 15.$

$$29. \quad 2x^2 + 5x - 17 + 3\sqrt{2x^2 + 5x - 17} = 28.$$

$$30. \quad 4x^2 + 7x + 5 - 3\sqrt{4x^2 + 7x + 5} = 4.$$

$$31. \quad 3x^2 - 2 = 10 + 3\sqrt{3x^2 - 2}.$$

$$32. \quad 2x^2 - 4x + 9 = 3 + 2\sqrt{2x^2 - 4x + 9}.$$

$$33. \quad x^2 + x = 4\sqrt{x^2 + x + 3} - 6.$$

$$34. \quad x^2 + x - 6 = 4\sqrt{x^2 + x + 6}.$$

$$35. \quad 3x^2 - 4x + 2\sqrt{3x^2 - 4x - 6} = 21.$$

$$36. \quad 16x - 3\sqrt{3x^2 - 16x + 21} = 3x^2 - 7.$$

$$37. \quad \sqrt{\frac{2x-5}{3x-7}} + \sqrt{\frac{3x-7}{2x-5}} = \frac{5}{2}.$$

$$38. \quad \frac{x^2 + x + 5}{x^2 + x - 2} + \frac{x^2 + x - 5}{x^2 + x - 4} = 10.$$

GRAPHIC SOLUTION OF QUADRATIC EQUATIONS*

330. Quadratic equations may be solved graphically by the method used in a preceding chapter. (§ 305.) It is, however, simpler to solve such equations by the method employed for the solution of simultaneous equations.

$$\text{Consider the equation} \quad ax^2 + bx + c = 0. \quad (1)$$

$$\text{Let} \quad y = x^2. \quad (2)$$

$$\text{Substituting in (1),} \quad ay + bx + c = 0. \quad (3)$$

The solution of the system (2), (3) for x produces the required root of (1).

The advantage of this method lies in the fact that the graph of the linear equation (3) is easily constructed, while the graph

* This section may be omitted.

of (2) is identical for all quadratic equations. Hence the graph of the equation $y = x^2$, which is represented in the annexed diagram, may be used for the solution of any numerical quadratic equation, provided its roots lie between the limits of the represented abscissas (-6 and $+6$).

Ex. 1. Solve $11x^2 + 30x - 165 = 0$. (1)

Let $y = x^2$. (2)

Then $11y + 30x - 165 = 0$. (3)

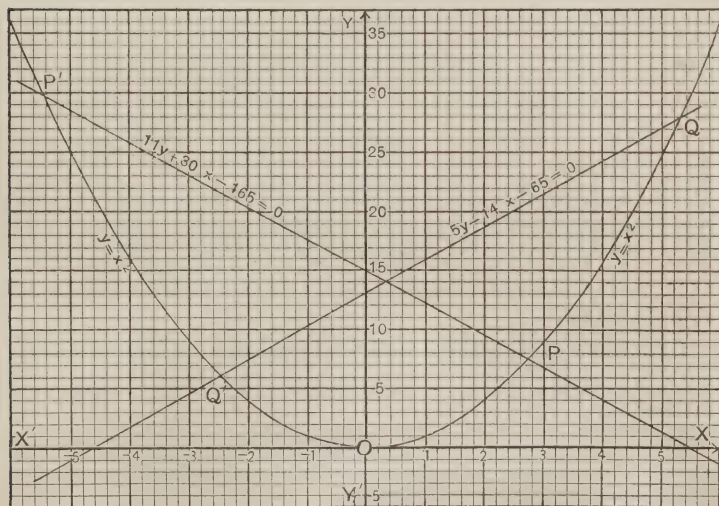
In (3), if $x = 0$, then $y = 15$; if $y = 0$, then $x = 5\frac{1}{2}$. The straight line joining the points $(0, 15)$ and $(5\frac{1}{2}, 0)$ is the graph of (3), which intersects the graph of (2) in P and P' . By measuring the abscissas of P and P' , we have

$$x = 2.7, \text{ or } x = -5.5.$$

Ex. 2. Solve $5x^2 - 14x - 65 = 0$. (1)

Let $y = x^2$. (2)

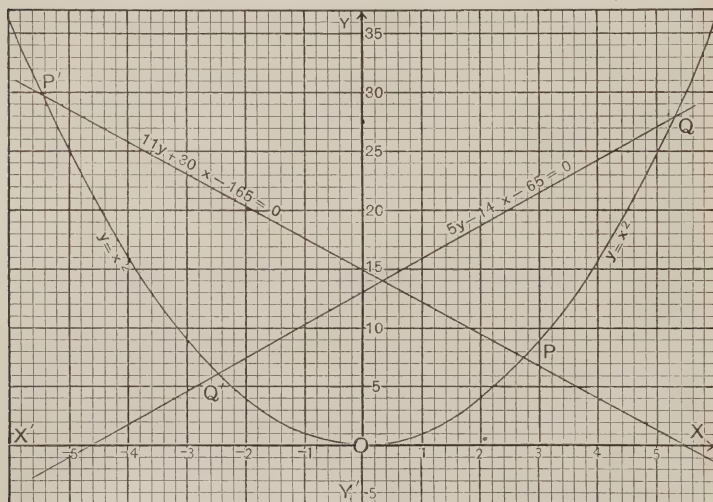
Then $5y - 14x - 65 = 0$. (3)



Locating two points of the equation (3), *e.g.* (0, 13) and (5, 27), and joining by a straight line produces the graph of (3), which intersects the graph of (2) in Q and Q' . Measuring the abscissas of Q and Q' , we obtain

$$x = 5.3, \text{ or } x = -2.5.$$

331. In the equation $ay + bx + c = 0$, if $x = 0$, then $y = -\frac{c}{a}$, and if $y = 0$, then $x = -\frac{c}{b}$. Hence, by taking on the x -axis the point $-\frac{c}{b}$ and on the y -axis the point $-\frac{c}{a}$, and applying a straight edge, the roots of the equation $ax^2 + bx + c = 0$ can frequently be determined by inspection. In some cases, how-



ever, it is more advantageous to locate one or both points outside the axes.

Equations of the third and higher degree, which are linear with the exception of one term, can be solved by the same method.

NOTE. The student should construct the graph of $y = x^2$ on a large scale. It is advisable to make the scale of the ordinates considerably smaller than the scale of the abscissas.

EXERCISE 125

Solve the following equations by the graphical method:

- | | |
|---|----------------------------|
| 1. $x^2 - x - 6 = 0.$ | 14. $x^2 - 2x - 9 = 0.$ |
| 2. $x^2 + x - 2 = 0.$ | 15. $3x^2 + 7x - 42 = 0.$ |
| 3. $x^2 - 3x - 18 = 0.$ | 16. $2x^2 + 5x - 20 = 0.$ |
| 4. $x^2 + 3x - 10 = 0.$ | 17. $x^2 + 2x - 10 = 0.$ |
| 5. $x^2 - 2x - 8 = 0.$ | 18. $5x^2 - 4x - 5 = 0.$ |
| 6. $x^2 + 2x - 4 = 0.$ | 19. $5x^2 + 4x - 25 = 0.$ |
| 7. $x^2 - 3x - 12 = 0.$ | 20. $9x^2 - 48x + 59 = 0.$ |
| 8. $x^2 + 3x - 10 = 0.$ | 21. $9x^2 - 12x - 26 = 0.$ |
| 9. $x^2 + x - 3 = 0.$ | 22. $9x^2 - 12x - 100 = 0$ |
| 10. $x^2 - 5x - 15 = 0.$ | 23. $3x^2 - 2x - 65 = 0.$ |
| 11. $4x^2 - 25x + 20 = 0.$ | 24. $2x^2 + x - 36 = 0.$ |
| 12. $3x^2 + 20x + 12 = 0.$ | 25. $2x^2 + 5x - 12 = 0.$ |
| 13. $x^2 + x - 5 = 0.$ | 26. $x^2 - 6x + 9 = 0.$ |
| 27. Has $x^2 + x + 1 = 0$ any real roots? | |
| 28. $3x^3 + 5x - 15 = 0.$ | 29. $x^3 - 5x - 2 = 0.$ |

CHAPTER XIX

SIMULTANEOUS QUADRATIC EQUATIONS

332. The degree of an equation involving several unknown quantities is equal to the greatest sum of the exponents of the unknown quantities contained in any term.

$xy + y = 4$ is of the second degree.

$x^3y + 5x^2y^3 - y^4$ is of the fifth degree.

333. A symmetrical equation is one which is not altered by interchanging the unknown quantities.

$x + y = xy$, $x^2 + x^4y^4 + y^2 = 4$ are symmetrical equations.

$x - y = 2$ and $x^3 - y^3 = 1$ are not symmetrical, but a change of sign would make them symmetrical.

334. A homogeneous equation is an equation all of whose terms are of the same degree with respect to the unknown quantities.

$4x^3 - 3x^2y = 3y^3$ and $x^2 - 2xy - 5y^2 = 0$ are homogeneous equations.

335. The absolute term of an equation is the term which does not contain any unknown quantity.

In $x^2 - 4xy + 2 = 0$ the absolute term is 2.

336. Simultaneous quadratic equations involving two unknown quantities lead, in general, to equations of the fourth degree. A few cases, however, can be solved by the methods of quadratics.*

* The graphic solution of simultaneous quadratic and higher equations has been treated in Chapter XVII.

I. EQUATIONS SOLVED BY FINDING $x + y$ AND $x - y$

337. If two of the quantities $x + y$, $x - y$, xy are given, the third one can be found by means of the relation $(x + y)^2 - 4xy = (x - y)^2$.

Ex. 1. Solve $\begin{cases} x + y = 5, \\ xy = 4. \end{cases}$ (1)

(2)

Squaring (1), $x^2 + 2xy + y^2 = 25$. (3)

(2) $\times 4$, $4xy = 16$. (4)

(3) $-$ (4), $x^2 - 2xy + y^2 = 9$.

Hence $x - y = \pm 3$. (5)

Combining (5) with (1), we have

$$\begin{array}{ll} x + y = 5, & \text{or} \quad x + y = 5, \\ x - y = 3. & x - y = -3. \end{array}$$

Hence $\begin{cases} x = 4, \\ y = 1. \end{cases}$ or $\begin{cases} x = 1, \\ y = 4. \end{cases}$

338. In many cases two of the quantities $x + y$, $x - y$, and xy are not given, but can be found.

Ex. 2. $\begin{cases} 2x^2 - 3xy + 2y^2 = 8, \\ x - y = 1. \end{cases}$ (1)

(2)

Square (2), $x^2 - 2xy + y^2 = 1$. (3)

(3) $\times 2$, $2x^2 - 4xy + 2y^2 = 2$. (4)

(1) $-$ (4), $xy = 6$.

Hence $4xy = 24$. (5)

(3) $+$ (5), $x^2 + 2xy + y^2 = 25$.

Therefore $x + y = 5$, or $x + y = -5$.

But $x - y = 1$, $x - y = 1$.

Hence $x = 3, y = 2$, $x = -2, y = -3$.

Check. $\begin{cases} 2 \cdot 3^2 - 3 \cdot 3 \cdot 2 + 2 \cdot 2^2 = 8, \\ 3 - 2 = 1. \end{cases}$ or $\begin{cases} 2 \cdot 2^2 - 3 \cdot 2 \cdot 3 + 2 \cdot 3^2 = 8, \\ -2 + 3 = 1. \end{cases}$

339. The roots of simultaneous quadratic equations must be arranged in pairs, *e.g.* the answers of the last example are:

$$\begin{cases} x = 3, \\ y = 2, \end{cases} \quad \text{or} \quad \begin{cases} x = -2, \\ y = -3. \end{cases}$$

EXERCISE 126

Solve:

- | | |
|---|---|
| 1. $\begin{cases} x + y = 5, \\ xy = 6. \end{cases}$ | 11. $\begin{cases} x^2 + y^2 = 25, \\ x - y = 1. \end{cases}$ |
| 2. $\begin{cases} x - y = 9, \\ xy = 36. \end{cases}$ | 12. $\begin{cases} x^2 + y^2 = 100, \\ xy = 48. \end{cases}$ |
| 3. $\begin{cases} x + y = 28, \\ xy = 187. \end{cases}$ | 13. $\begin{cases} x^2 + y^2 = 82, \\ x + y = -8. \end{cases}$ |
| 4. $\begin{cases} x - y = 5, \\ xy = 176. \end{cases}$ | 14. $\begin{cases} x^2 + xy + y^2 = 7, \\ x - y = 1. \end{cases}$ |
| 5. $\begin{cases} x - y = 24, \\ xy = 4212. \end{cases}$ | 15. $\begin{cases} x^2 - 3xy + y^2 = -11, \\ x^2 + y^2 = 34. \end{cases}$ |
| 6. $\begin{cases} x + y = -6, \\ xy = -2592. \end{cases}$ | 16. $\begin{cases} x + y + xy = 11, \\ x + y = 6. \end{cases}$ |
| 7. $\begin{cases} x - y = 61, \\ xy = 876. \end{cases}$ | 17. $\begin{cases} \frac{1}{x} + \frac{1}{y} = 3, \\ x + y = 1\frac{1}{2}. \end{cases}$ |
| 8. $\begin{cases} x^2 + y^2 = 50, \\ xy = 7. \end{cases}$ | 18. $\begin{cases} x^2 + 3xy + y^2 = 211, \\ \frac{x+y}{13} = 1. \end{cases}$ |
| 9. $\begin{cases} x^2 + y^2 = 37, \\ x + y = 7. \end{cases}$ | 19. $\begin{cases} x - y = b, \\ xy = a^2 + ab. \end{cases}$ |
| 10. $\begin{cases} x^2 + y^2 = 13, \\ x + y = 5. \end{cases}$ | 20. $\begin{cases} x + y = 2a, \\ x^2 + y^2 = 2a^2 + 2b^2. \end{cases}$ |

II. ONE EQUATION LINEAR, THE OTHER QUADRATIC

340. A system of simultaneous equations, one linear and one quadratic, can be solved by eliminating one of the unknown quantities by means of substitution.

Ex. Solve $2x + 3y = 7$, (1)

$x^2 + 2y^2 - y = 5$. (2)

From (1) we have, $x = \frac{7-3y}{2}$. (3)

Substituting in (2), $\left(\frac{7-3y}{2}\right)^2 + 2y^2 - y = 5$.

Simplifying, $49 - 42y + 9y^2 + 8y^2 - 4y = 20$.

Transposing, etc., $17y^2 - 46y + 29 = 0$.

Factoring, $(y-1)(17y-29) = 0$.

Hence $y = 1$, or $\frac{29}{17}$.

Substituting in (3), $x = 2$, or $\frac{17}{17}$.

EXERCISE 127

Solve:

1. $\begin{cases} xy - 5x = 1, \\ 7x - y = 1. \end{cases}$

7. $\begin{cases} x^2 + y^2 - 40 = 0, \\ 3x - y = 0. \end{cases}$

2. $\begin{cases} xy = 6, \\ 2x + 3y = 13. \end{cases}$

8. $\begin{cases} x - 3y + 5 = 0, \\ y - xy = 0. \end{cases}$

3. $\begin{cases} x^2 + 4xy = 57, \\ x + y = 7. \end{cases}$

9. $\begin{cases} x^2 + y^2 = 37, \\ 9x + 7y = 61. \end{cases}$

4. $\begin{cases} x - y = 6, \\ xy = 16. \end{cases}$

10. $\begin{cases} x^2 + y^2 = 7 + xy, \\ 2x - 3y = 0. \end{cases}$

5. $\begin{cases} 2x - 3y = 1, \\ 5x^2 - 7y^2 = 13. \end{cases}$

11. $\begin{cases} x^2 + 4xy = 96, \\ x + y = 9. \end{cases}$

6. $\begin{cases} 5x^2 + y = 3xy, \\ 2x - y = 0. \end{cases}$

12. $\begin{cases} \frac{5x-y}{4} = \frac{7}{4x+3y}, \\ 3x - 2y = 1. \end{cases}$

$$13. \quad \begin{cases} \frac{x}{3} + \frac{y}{4} = 2, \\ \frac{3}{x} + \frac{4}{y} = \frac{3}{2}. \end{cases}$$

$$14. \quad \begin{cases} \frac{x}{4} + \frac{y}{5} = 3, \\ (x+y)^2 = 200 - x. \end{cases}$$

$$15. \quad \begin{cases} x:y = 2:3, \\ x^2 + y^2 = 5(x+y) + 2. \end{cases}$$

$$16. \quad \begin{cases} \frac{y+8}{x+2} = \frac{3x+y}{3x-y}, \\ 2x - y = 2. \end{cases}$$

III. HOMOGENEOUS EQUATIONS

341. If one equation of two simultaneous quadratics is homogeneous, the example can always be reduced to an example of the preceding type, for one unknown quantity can be expressed in terms of the other.

Consider the homogeneous equation,

$$4x^2 - 11xy + 6y^2 = 0. \quad (1)$$

Expressing x in terms of y by means of the formula for quadratics,

$$\begin{aligned} x &= \frac{11y \pm \sqrt{(11y)^2 - 4 \cdot 4 \cdot 6y^2}}{8} \\ &= \frac{11y \pm 5y}{8} \\ &= 2y, \text{ or } \frac{3}{4}y. \end{aligned}$$

In most cases this result can be obtained more simply by factoring, *e.g.* factor (1),

$$(x - 2y)(4x - 3y) = 0.$$

Hence $x - 2y = 0, 4x - 3y = 0.$

Combining these results with another quadratic equation produces two systems of the preceding kind.

$$\text{Ex. 1. Solve } \begin{cases} x^2 - 3y^2 + 2y = 3, \\ 2x^2 - 7xy + 6y^2 = 0. \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Factor (2), $(x - 2y)(2x - 3y) = 0.$

Hence we have to solve the two systems

$$\begin{cases} x - 2y = 0, \\ x^2 - 3y^2 + 2y = 3. \end{cases}$$

From (3), $x = 2y$.

Substituting in (1),

$$4y^2 - 3y^2 + 2y = 3,$$

$$y^2 + 2y - 3 = 0,$$

$$(y - 1)(y + 3) = 0.$$

$$\text{Hence } \begin{cases} y = 1, \\ x = 2. \end{cases} \quad \begin{cases} -3, \\ -6. \end{cases}$$

$$\text{or } \begin{cases} 2x - 3y = 0, \\ x^2 - 3y^2 + 2y = 3. \end{cases} \quad \begin{matrix} (3) \\ (1) \end{matrix}$$

$$x = \frac{3}{2}y.$$

$$\frac{9y^2}{4} - 3y^2 + 2y = 3,$$

$$3y^2 - 8y + 12 = 0,$$

$$y = \frac{8 \pm \sqrt{-80}}{6}.$$

$$y = \frac{4 + 2\sqrt{-5}}{3}, \quad \frac{4 - 2\sqrt{-5}}{3}.$$

$$x = 2 + \sqrt{-5}, \quad 2 - \sqrt{-5}.$$

342. If both equations are homogeneous with exception of the absolute term, the problem can be reduced to the preceding case by eliminating the absolute term.

$$\text{Ex. 2. Solve } \begin{cases} 3x^2 - 4xy + 3y^2 = 2, \\ 2x^2 - 2xy + 5y^2 = 5. \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Eliminate 2 and 5 by subtraction.

$$(1) \times 5, \quad 15x^2 - 20xy + 15y^2 = 2 \times 5. \quad (3)$$

$$(2) \times 2, \quad 4x^2 - 4xy + 10y^2 = 5 \times 2. \quad (4)$$

$$\text{Subtracting,} \quad 11x^2 - 16xy + 5y^2 = 0.$$

$$\text{Factoring,} \quad (x - y)(11x - 5y) = 0.$$

Hence solve :

$$\begin{cases} x - y = 0, \\ 2x^2 - 2xy + 5y^2 = 5. \end{cases}$$

From (3), $x = y$.

Substituting y in (2),

$$2y^2 - 2y^2 + 5y^2 = 5,$$

$$y^2 = 1,$$

$$y = \pm 1,$$

$$x = \pm 1.$$

$$\text{or } \begin{cases} 11x - 5y = 0, \\ 2x^2 - 2xy + 5y^2 = 5. \end{cases} \quad \begin{matrix} (5) \\ (2) \end{matrix}$$

$$y = \frac{11}{5}x,$$

$$2x^2 - 2 \cdot \frac{11}{5}x^2 + 5 \cdot \frac{121}{25}x^2 = 5,$$

$$\frac{109x^2}{5} = 5,$$

$$x^2 = \frac{25}{109},$$

$$x = \pm \frac{5}{\sqrt{109}}, \quad y = \pm \frac{11}{\sqrt{109}}.$$

343. In general two quadratic equations which are homogeneous with the exception of one term can be solved if these terms are similar. For the similar terms can be eliminated.

To solve
$$\begin{cases} 4x^2 - 9xy + y^2 - 4y = 0, \\ 2x^2 - 2xy - 4y^2 - 3y = 0. \end{cases}$$

Eliminate the terms containing the first powers of y , and proceed as before.

EXERCISE 128

Solve:

- | | |
|---|---|
| 1. $\begin{cases} x^2 - 9y^2 = 7, \\ xy + 4y^2 = 8. \end{cases}$ | 11. $\begin{cases} x^2 - 2xy = 5, \\ x^2 - y^2 = 21. \end{cases}$ |
| 2. $\begin{cases} x + xy = 6, \\ 7x^2 + 2xy - 5y^2 = 0. \end{cases}$ | 12. $\begin{cases} 2x^2 - 2xy + y^2 = 2, \\ x^2 - y^2 = -3. \end{cases}$ |
| 3. $\begin{cases} x^2 - y^2 = 8, \\ x^2 - 4xy + 3y^2 = 0. \end{cases}$ | 13. $\begin{cases} 2x^2 - 3xy + y^2 = 3, \\ x^2 + 2xy - 3y^2 = 5. \end{cases}$ |
| 4. $\begin{cases} 3x^2 - xy - 2y^2 = 0, \\ 4x^2 - 9xy + 2y^2 = -3. \end{cases}$ | 14. $\begin{cases} 3xy + y^2 = 28, \\ 4x^2 + xy = 8. \end{cases}$ |
| 5. $\begin{cases} 2x^2 - 3xy + y^2 = 0, \\ x^2 + 4xy + 2y^2 = 17. \end{cases}$ | 15. $\begin{cases} x^2 - xy + y^2 = 39, \\ 2x^2 - 3xy + 2y^2 = 43. \end{cases}$ |
| 6. $\begin{cases} 2x^2 - xy - 6y^2 = 9, \\ 3x^2 - 10xy + 3y^2 = 0. \end{cases}$ | 16. $\begin{cases} 6x^2 - 4xy - 8 = 0, \\ 2xy + 5y^2 - 28 = 0. \end{cases}$ |
| 7. $\begin{cases} 4x^2 = 9xy - 5y^2, \\ 7x^2 - 3xy - 3x - 2y = 1. \end{cases}$ | 17. $\begin{cases} 2x^2 - 3xy + y^2 = 3, \\ x^2 + 2y^2 - 6 = 0. \end{cases}$ |
| 8. $\begin{cases} x^2 + 6y^2 = 5xy, \\ x^2 + 2y^2 - 3x + 2y = 2. \end{cases}$ | 18. $\begin{cases} x^2 + 2xy + y^2 = 9y, \\ 2x^2 - 3xy + 2y^2 = 4y. \end{cases}$ |
| 9. $\begin{cases} 3xy + y^2 = 7, \\ x^2 + xy = 6. \end{cases}$ | 19. $\begin{cases} 3x^2 - 3xy + 2y^2 = 2x, \\ 2x^2 + 3y^2 - 4x = xy. \end{cases}$ |
| 10. $\begin{cases} x^2 - y^2 = 16, \\ 2x^2 - 4xy + 3y^2 = 17. \end{cases}$ | |

IV. SPECIAL DEVICES

344. Many examples belonging to the preceding types, and others not belonging to them, can be solved by special devices.

345. A. Symmetrical equations, and equations which would be symmetrical if a sign were changed, can often be solved by the substitution $x = u + v$, $y = u - v$.

$$\text{Ex. 1. Solve} \quad \begin{cases} x^4 + y^4 = 2, \\ x - y = 2. \end{cases} \quad (1)$$

Let $x = u + v$, $y = u - v$.

Substituting in (1), $(u + v)^4 + (u - v)^4 = 2$,

$$\begin{aligned} \text{or} \quad & u^4 + 4u^3v + 6u^2v^2 + 4uv^3 + v^4, \\ & + u^4 - 4u^3v + 6u^2v^2 - 4uv^3 + v^4 = 2, \\ \text{or} \quad & 2u^4 + 12u^2v^2 + 2v^4 = 2. \end{aligned}$$

$$\text{Dividing by 2,} \quad u^4 + 6u^2v^2 + v^4 = 1. \quad (3)$$

$$\begin{aligned} \text{Similarly from (2),} \quad & 2v = 2, \\ & v = 1. \end{aligned} \quad (4)$$

$$\text{Substituting } v \text{ in (3),} \quad u^4 + 6u^2 + 1 = 1,$$

$$\text{or} \quad u^4 + 6u^2 = 0.$$

$$\text{Factoring,} \quad u^2(u^2 + 6) = 0.$$

$$\text{Therefore} \quad u = 0, \quad u = \pm \sqrt{-6}.$$

$$\text{Hence} \quad \begin{aligned} x &= \pm \sqrt{-6} + 1, \\ y &= \pm \sqrt{-6} - 1. \end{aligned} \quad \left. \begin{array}{l} \\ \end{array} \right\} \begin{array}{l} +1, \\ -1. \end{array}$$

346. B. Equations of higher degree can sometimes be reduced to equations of the second degree by **dividing member by member**.

$$\text{Ex. Solve} \quad \begin{cases} x^3 + y^3 = 28, \\ x + y = 4. \end{cases} \quad (1)$$

$$\text{Dividing (1) by (2),} \quad x^2 - xy + y^2 = 7. \quad (3)$$

$$\text{Squaring (2),} \quad x^2 + 2xy + y^2 = 16. \quad (4)$$

$$\begin{aligned} (4) - (3), \quad & 3xy = 9, \\ & xy = 3. \end{aligned} \quad (5)$$

$$(3) - (5). \quad \begin{aligned} x^2 - 2xy + y^2 &= 4, \\ x - y &= \pm 2. \end{aligned}$$

$$\text{Combining with (2),} \quad \begin{cases} x = 3, \\ y = 1. \end{cases} \text{ or } \begin{cases} x = 1, \\ y = 3. \end{cases}$$

347. C. Some simultaneous quadratics can be solved by considering not x or y but some other quantity, as $\frac{1}{x}$, xy , x^2 , $(x+y)$, x^2y , etc., at first as the unknown quantity.

$$\text{Ex. 1. Solve } \begin{cases} x + y + \sqrt{x+y} = 20, \\ x - y - \sqrt{x-y} = 6. \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Considering $\sqrt{x+y}$ and $\sqrt{x-y}$ as unknown quantities and solving, we have

$$\text{from (1),} \quad \sqrt{x+y} = 4 \text{ or } -5,$$

$$\text{from (2),} \quad \sqrt{x-y} = 3 \text{ or } -2.$$

But the negative roots being extraneous, we obtain by squaring,

$$x + y = 16,$$

$$x - y = 9.$$

$$\text{Therefore} \quad x = 12\frac{1}{2}, \quad y = 3\frac{1}{2}.$$

$$\text{Ex. 2. Solve } \begin{cases} x^2y^2 + xy - 12 = 0, \\ x + y = 4. \end{cases} \quad \begin{matrix} (1) \\ (2) \end{matrix}$$

Solving (1) for xy , $xy = 3$ or -4 .

$$\text{Combining with (2),} \quad \begin{cases} xy = 3, \\ x + y = 4. \end{cases} \text{ or } \begin{cases} xy = -4, \\ x + y = 4. \end{cases}$$

$$\text{Solving according to § 337,} \quad x - y = 2. \quad x - y = \pm 4\sqrt{2}.$$

$$\begin{aligned} &\left. \begin{array}{l} x = 3, \\ y = 1. \end{array} \right\} \quad \left. \begin{array}{l} x = 1, \\ y = 3. \end{array} \right\} \quad \left. \begin{array}{l} x = 2 + 2\sqrt{2}, \\ y = 2 - 2\sqrt{2}. \end{array} \right\} \quad \left. \begin{array}{l} x = 2 - 2\sqrt{2}, \\ y = 2 + 2\sqrt{2}. \end{array} \right\} \end{aligned}$$

$$\text{Ex. 3. Solve } x^2 + 4y^2 - 2x + 4y = 27, \quad (1)$$

$$xy = 6. \quad (2)$$

Multiplying (2) by 4 and subtracting from (1),

$$x^2 - 4xy + 4y^2 - 2x + 4y = 3,$$

$$\text{or} \quad (x - 2y)^2 - 2(x - 2y) - 3 = 0. \quad (3)$$

Considering $(x - 2y)$ as unknown quantity and solving (3),

$$x - 2y = 3, \text{ or } -1.$$

Hence we have to solve the two systems

$$\begin{cases} x - 2y = -1, \\ xy = 6. \end{cases} \quad \begin{cases} x - 2y = 3, \\ xy = 6. \end{cases}$$

The solution produces the roots :

$$\begin{aligned} x = 3, & \left\{ \begin{array}{l} -4, \\ \frac{+3 + \sqrt{57}}{2}, \end{array} \right\} \quad \left\{ \begin{array}{l} \frac{+3 - \sqrt{57}}{2}, \\ \frac{-3 - \sqrt{57}}{4} \end{array} \right\} \\ y = 2, & \left\{ \begin{array}{l} -\frac{3}{2}, \\ \frac{-3 + \sqrt{57}}{4} \end{array} \right\} \quad \left\{ \begin{array}{l} \frac{-3 - \sqrt{57}}{4} \end{array} \right\} \end{aligned}$$

348. D. If the quadratic terms can be eliminated, the example can be reduced to one of type II (one equation linear, one quadratic).

$$\text{Ex. 1. Solve } \begin{cases} 2xy - x + 2y = 16, \\ 3xy + 2x - 4y = 10. \end{cases} \quad (1)$$

$$(2)$$

$$(1) \times 3, \quad 6xy - 3x + 6y = 48. \quad (3)$$

$$(2) \times 2, \quad 6xy + 4x - 8y = 20. \quad (4)$$

$$(4) - (3), \quad 7x - 14y = -28.$$

$$\text{Dividing by 7,} \quad x - 2y = -4.$$

$$\text{Hence,} \quad x = 2y - 4.$$

$$\text{Substituting in (1), } 2y(2y - 4) - (2y - 4) + 2y = 16.$$

$$\text{Solving,} \quad y = 3, \left\{ \begin{array}{l} -1, \end{array} \right\}$$

$$\text{Substituting in (1), } x = 2. \left\{ \begin{array}{l} -6. \end{array} \right\}$$

EXERCISE 129

Solve by the method of symmetrical equations :

$$1. \quad \begin{cases} x^4 + y^4 = 17, \\ x + y = 3. \end{cases}$$

$$3. \quad \begin{cases} x^3 - y^3 = 19, \\ x - y = 1. \end{cases}$$

$$2. \quad \begin{cases} x^4 + y^4 = 337, \\ x - y = 1. \end{cases}$$

$$4. \quad \begin{cases} x^4 + y^4 = 82, \\ x + y = 4. \end{cases}$$

Solve by dividing member by member :

$$5. \begin{cases} x^3 + y^3 = 72, \\ x + y = 6. \end{cases}$$

$$8. \begin{cases} x^4 + x^2 y^2 + y^4 = 21, \\ x^2 + xy + y^2 = 7. \end{cases}$$

$$6. \begin{cases} x^3 - y^3 = 217, \\ x - y = 1. \end{cases}$$

$$9. \begin{cases} x^3 - y^3 = 37, \\ x - y = 1. \end{cases}$$

$$7. \begin{cases} x^4 - y^4 = 80, \\ x^2 - y^2 = 8. \end{cases}$$

$$10. \begin{cases} x^4 + x^2 y^2 + y^4 = 3, \\ x^2 - xy + y^2 = 1. \end{cases}$$

Solve by the method of § 347 :

$$11. \begin{cases} 5x^2 + 2y^2 = 22, \\ 3x^2 - 5y^2 = 7. \end{cases}$$

$$13. \begin{cases} x + y = 25, \\ \sqrt{x} + \sqrt{y} = 7. \end{cases}$$

$$12. \begin{cases} x^2 y^2 + 6 = 5xy, \\ x^2 + 4y^2 = 5. \end{cases}$$

$$14. \begin{cases} \sqrt{\frac{x+y}{x}} + \sqrt{\frac{x}{x+y}} = 4\frac{1}{4}, \\ x^2 + 2y^2 = 76. \end{cases}$$

Solve by eliminating the quadratic terms :

$$15. \begin{cases} xy + x = 15, \\ xy + y = 16. \end{cases}$$

$$16. \begin{cases} 4x^2 - x + y = 67, \\ 3x^2 - 3y = 27. \end{cases}$$

$$17. \begin{cases} 4x^2 - 5y^2 - 3x + y + 3 = 0, \\ 8x^2 - 10y^2 - 7x + 9y = 0. \end{cases}$$

Solve by any method :

$$18. \begin{cases} x^2 + y^2 = 1274, \\ x = 5y. \end{cases}$$

$$22. \begin{cases} x + 2y = 8, \\ x^2 + y^2 = 13. \end{cases}$$

$$19. \begin{cases} x^3 + y^3 = 16,021, \\ x + y = 37. \end{cases}$$

$$23. \begin{cases} 3x + 8y = 20, \\ x^2 + 2y^2 + xy = 5x + y + 1 \end{cases}$$

$$20. \begin{cases} x^2 + xy = 150, \\ y^2 + xy = 75. \end{cases}$$

$$24. \begin{cases} 3x^2 - 10xy + 3y^2 = 0, \\ 3x^2 - y = 26. \end{cases}$$

$$21. \begin{cases} x^2 + xy - y^2 = -31, \\ x^2 - xy + y^2 = 49. \end{cases}$$

$$25. \begin{cases} x + 5y = 3, \\ x^2 + 3xy - 2y^2 + 1 = 0. \end{cases}$$

$$26. \begin{cases} 5xy - 9y^2 = 1, \\ x^2 - 2xy = 0. \end{cases}$$

$$27. \begin{cases} x^4 - y^4 = 65, \\ x^2 - y^2 = 5. \end{cases}$$

$$28. \begin{cases} x^2 - y^2 = x + y, \\ x + 2y = 7. \end{cases}$$

$$29. \begin{cases} x^2 + xy + y^2 = 37, \\ x^2 + y^2 = 25. \end{cases}$$

$$30. \begin{cases} x^2 + 3xy + 2y^2 = 20, \\ 4x^2 + 5y^2 - 41 = 0. \end{cases}$$

$$31. \begin{cases} 2x^2 - 3y^2 = 6, \\ 3x^2 - 2y^2 = 19. \end{cases}$$

$$32. \begin{cases} x^2 - 40 = -y^2, \\ \frac{x}{y} = 3. \end{cases}$$

$$33. \begin{cases} xy = 12, \\ 2x + 3y = 18. \end{cases}$$

$$34. \begin{cases} 5x^2 + y = 3xy, \\ 2x - y = 0. \end{cases}$$

$$35. \begin{cases} 5x^2 + 2y^2 = 22, \\ 3x^2 - 5y^2 = 7. \end{cases}$$

$$36. \begin{cases} x^2 + y^2 + xy = 147, \\ x + y = 13. \end{cases}$$

$$37. \begin{cases} x^2 + y^2 = 130, \\ \frac{x+y}{x-y} = 8. \end{cases}$$

$$38. \begin{cases} 4x^2 + 3xy + 2y^2 = 18, \\ 3x^2 + 2xy - y^2 = 3. \end{cases}$$

$$39. \begin{cases} (3x-2y)(2x-3y)=26, \\ x+1=2y. \end{cases}$$

$$40. \begin{cases} x^2 - 2xy - y^2 = 1, \\ 3x^2 - 4xy = 35. \end{cases}$$

$$41. \begin{cases} 3x^2 + 4xy - y^2 = 14, \\ 2x^2 + 5xy + 6y^2 = 9. \end{cases}$$

$$42. \begin{cases} x + y + \sqrt{x+y} = 12, \\ x^2 + y^2 = 45. \end{cases}$$

$$43. \begin{cases} x^4 + y^4 = 97, \\ x + y = 5. \end{cases}$$

$$44. \begin{cases} x^5 - y^5 = 242, \\ x - y = 2. \end{cases}$$

$$45. \begin{cases} \frac{2x-y+1}{x-2y+1} = \frac{3}{8}, \\ x^2 - 3xy + y^2 = 5. \end{cases}$$

$$46. \begin{cases} x:y = 1:9, \\ x:6 = 6:y. \end{cases}$$

$$47. \begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{5}{6}, \\ \frac{1}{x^2} + \frac{1}{y^2} = \frac{13}{36}. \end{cases}$$

$$48. \begin{cases} \frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{4}, \\ \frac{1}{x} - \frac{1}{y} = \frac{1}{2}. \end{cases}$$

$$49. \begin{cases} \frac{1}{x^3} + \frac{1}{y^3} = 35, \\ \frac{1}{x} + \frac{1}{y} = 5. \end{cases}$$

$$50. \begin{cases} \frac{1}{x^3} - \frac{1}{y^3} = 63, \\ \frac{1}{x} - \frac{1}{y} = 3. \end{cases} \quad 53. \begin{cases} \frac{x\sqrt{x} + y\sqrt{y}}{x\sqrt{x} - y\sqrt{y}} = \frac{35}{19}, \\ x + y = 13. \end{cases}$$

$$51. \begin{cases} x(x^3 + y^3) = 84, \\ y(x^3 + y^3) = 28. \end{cases} \quad 54. \begin{cases} x + y = 520, \\ \sqrt[3]{x} + \sqrt[3]{y} = 10. \end{cases}$$

$$52. \begin{cases} (x + y)(x^2 + y^2) = 175, \\ (x - y)(x^2 + y^2) = 25. \end{cases} \quad 55. \begin{cases} \frac{1}{x} + \frac{1}{y} = \frac{1}{3}, \\ x^2 + y^2 = 160. \end{cases}$$

$$56. \begin{cases} (x + y)^2 = 3x^2 - 2, \\ (x - y)^2 = 3y^2 - 11. \end{cases}$$

$$57. \begin{cases} x^2 - 2xy + 3y^2 - 3(x - y) = 0, \\ 2x^2 + xy - y^2 - 9(x - y) = 0. \end{cases}$$

$$58. \begin{cases} 2x^2 - 3xy = 9(x - 2y), \\ x^2 - 3y^2 = 6(x - 2y). \end{cases}$$

$$59. \begin{cases} (x + 2y)(x + 3y) = 3(x + y), \\ (2x + y)(3x + y) = 28(x + y). \end{cases}$$

$$60. \begin{cases} x^2 + 2xy + y^2 - 3 = 2(x + y), \\ xy = 2. \end{cases}$$

$$61. \begin{cases} \sqrt[3]{x^4} + \sqrt[5]{y^2} = 20, \\ \sqrt[3]{x^2} + \sqrt[5]{y} = 6. \end{cases}$$

$$66. \begin{cases} (x^2 + y^2)(x^3 + y^3) = 455, \\ x + y = 5. \end{cases}$$

$$62. \begin{cases} x^2 - xy + y^2 = 13, \\ x^3 + y^3 = 91. \end{cases}$$

$$67. \begin{cases} x + y = m, \\ x^2 + y^2 = n. \end{cases}$$

$$63. \begin{cases} x^3 + y^3 = 133, \\ x^2 - xy + y^2 = 19. \end{cases}$$

$$68. \begin{cases} \frac{x+y}{y} = a, \\ xy = b. \end{cases}$$

$$64. \begin{cases} x^2 + y^2 + x + y = 18, \\ xy = 6. \end{cases}$$

$$65. \begin{cases} x^2 + y^2 + x - y = 6, \\ (x^2 + y^2)(x - y) = 5. \end{cases}$$

$$69. \begin{cases} \frac{x+y}{x-y} = \frac{a^2 + b^2}{a^2 - b^2}, \\ xy = b^2. \end{cases}$$

$$\begin{array}{ll}
70. \quad \begin{cases} x^2 - xy = ay^2, \\ xy - y^2 = b^2x. \end{cases} & 71. \quad \begin{cases} (2a - x)^2 - (2b - y)^2 = a^2 - b^2, \\ (2a - x)(2b - y) = ab. \end{cases} \\
72. \quad \begin{cases} x^2 - 3xy + 4z^2 = 31, \\ 3x + y = 5, \\ 2x - 3z = -7. \end{cases} & 73. \quad \begin{cases} x(x + y + z) = a, \\ y(x + y + z) = b, \\ z(x + y + z) = c. \end{cases}
\end{array}$$

EXERCISE 130

PROBLEMS

1. The sum of two numbers is 44, and the sum of their squares is 1000. Find the numbers.
2. The difference of two numbers is 2, and the sum of their squares exceeds their product by 103. Find the numbers.
3. The sum of the squares of two numbers is 40, and the product of the numbers is equal to three times their difference.
4. The hypotenuse of a right triangle is 13, and the sum of the other two sides is 17. Find these sides. (§ 319.)
5. The area of a right triangle is 330 square feet, and the hypotenuse is 61 feet. Find the other two sides.
6. The hypotenuse exceeds one side of a right triangle by 2 inches, and the third side is 6 inches. Find the unknown sides of the triangle.
7. The perimeter of a right triangle is 70 feet, and its area is 210 square feet. Find the three sides.
8. The mean proportional of two numbers is 10, and the sum of their squares is 2504. Find the numbers.
9. To inclose a rectangular field 68,200 square feet in area, 1460 feet of fence are required. Find the dimensions of the field.
10. The area of a rectangle is 1008 square feet, and the diagonal (Ex. 23, p. 303) is 65 feet. Find the length of the sides.

11. The sum of the radii of two circles is 113 centimeters and their areas are together equal to the area of a circle whose radius is 85 centimeters. Find the radii. (Area of circle $= \pi R^2$.)

12. The radii of two spheres differ by 18 inches, and the difference of the spherical surfaces is equal to a sphere whose radius is 48 inches. Find the radii. (Surface of sphere $= 4\pi R^2$.)

13. A dealer sells a number of horses for \$1320, receiving the same price for each animal. If he had sold one horse less, but charged \$10 apiece more, he would have received the same sum. Find the price of a horse.

14. Two cubes together contain $23\frac{5}{8}$ cubic inches, and the edge of one, increased by the edge of the other, equals $4\frac{1}{2}$ inches. Find the edge of each cube.

15. The volumes of two cubes differ by 61 cubic centimeters, and the edge of the larger exceeds the edge of the smaller by 1 centimeter. Find their edges.

16. A number less than 100 is equal to four times the sum of its digits, and the sum of the squares of the digits is 20. Find the number.

17. The sum of the squares of two numbers added to the difference of the numbers equals 292. The same sum multiplied by the difference of the numbers equals 3091. What are the numbers?

CHAPTER XX

PROPERTIES OF QUADRATIC EQUATIONS

CHARACTER OF THE ROOTS

349. The quadratic equation $ax^2 + bx + c = 0$ has two roots,

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad (\S 322.)$$

Hence it follows:

1. *If $b^2 - 4ac$ is positive or equal to zero, the roots are real.
If $b^2 - 4ac$ is negative, the roots are imaginary.*
2. *If $b^2 - 4ac$ is a perfect square, the roots are rational.
If $b^2 - 4ac$ is not a perfect square, the roots are irrational.*
3. *If $b^2 - 4ac$ is zero, the roots are equal.
If $b^2 - 4ac$ is not zero, the roots are unequal.*

350. The expression $b^2 - 4ac$ is called the **discriminant** of the equation $ax^2 + bx + c = 0$.

Ex. 1. Determine the character of the roots of the equation $3x^2 - 2x - 5 = 0$.

$$\text{The discriminant} = (-2)^2 - 4 \cdot 3 \cdot (-5) = 64.$$

Hence the roots are real, rational, and unequal.

Ex. 2. Determine the character of the roots of the equation $4x^2 - 12x + 9 = 0$.

$$\text{Since } (-12)^2 - 4 \cdot 4 \cdot 9 = 0, \text{ the roots are real, rational, and equal.}$$

Ex. 3. Prove that the roots of the equation $x^2 + 2px + p^2 - q^2 - 2qr - r^2 = 0$ are rational.

$$\begin{aligned} \text{The discriminant} &= (2p)^2 - 4(p^2 - q^2 - 2qr - r^2), \\ &= 4(q^2 + 2qr + r^2), \\ &= 4(q + r)^2. \end{aligned}$$

Hence the roots are rational.

351. The preceding propositions make it possible to determine the coefficients so that the roots shall satisfy a given condition.

Ex. 1. Determine the value of m for which the roots of the equation $-\frac{1}{4}x^2 + \frac{3}{2}x + 3 = m$ are equal.

Transposing, $-\frac{1}{4}x^2 + \frac{3}{2}x + (3 - m) = 0.$

The discriminant must equal zero. (§ 349.)

Hence $(\frac{3}{2})^2 - 4(-\frac{1}{4})(3 - m) = 0.$

Or $\frac{9}{4} + 3 - m = 0.$

Hence $m = 5\frac{1}{4}.$

NOTE. This result can be obtained by inspection of the graph of this function, which was discussed in § 302.

Ex. 2. Determine the value of m for which the equation $(m + 5)x^2 + 3mx - 4(m - 5) = 0$ has equal roots.

The discriminant must equal zero.

Hence $(3m)^2 + 4 \cdot 4(m + 5)(m - 5) = 0.$

Or $25m^2 - 400 = 0.$

Therefore $m = +4, \text{ or } -4.$

Check. The equations $9x^2 + 12x + 4 = 0$, and $x^2 - 12x + 36 = 0$, have equal roots.

EXERCISE 131

Determine, without solution, the character of the roots of each equation:

1. $x^2 - 7x + 12 = 0.$

10. $x^2 = \frac{1}{5}x - 1.$

2. $3x^2 - 10x + 3 = 0.$

11. $x^2 + 15x = 17.$

3. $9x^2 - 6x + 1 = 0.$

12. $9x^2 = 24x - 16.$

4. $x^2 + 2x + 9723 = 0.$

13. $9x^2 = 17x - 47,125.$

5. $x^2 - 7x - 7 = 0.$

14. $15 + \frac{4}{x} = \frac{17}{x^2}.$

6. $2x^2 + x + 1 = 0.$

15. $\frac{x-8}{x-3} = x.$

7. $6x^2 - 5x - 1 = 0.$

8. $x^2 = x + 12.$

9. $x^2 - x\sqrt{2} = 3.$

16. $(x+4)(x+13) = 90.$

Determine the value of m for which the roots of the following equations are equal :

17. $2x^2 - 8x + m = 0.$

22. $4x^2 - 2x = 3m.$

18. $3x^2 + 6x + m = 0.$

23. $x^2 - 6x + m = 0.$

19. $5x^2 + 20x = m.$

24. $x^2 - (m+3)x + 3m = 0.$

20. $mx^2 - 14x - 7 = 0.$

25. $x^2 + (m+5)x + 5m = 0.$

21. $x^2 + 2x + 3 = m.$

26. $(m+1)x^2 + 3mx = 1 - m.$

27. $x^2 - (2m-3)x + 2m = 0.$

28. $(13+3m)x^2 + 5mx + 13 - 3m = 0.$

RELATION BETWEEN ROOTS AND COEFFICIENTS

352. If the roots of the equation $ax^2 + bx + c = 0$ are denoted by r_1 and r_2 , then

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Hence

$$r_1 + r_2 = -\frac{b}{a},$$

and

$$r_1 r_2 = \frac{(-b)^2 - (b^2 - 4ac)}{4a^2}.$$

Or

$$r_1 r_2 = \frac{c}{a}.$$

If the given equation is written in the form $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, these results may be expressed as follows:

353. *If the coefficient of x^2 in a quadratic equation is unity,*

(a) *The sum of the roots is equal to the coefficient of x with the sign changed.*

(b) *The product of the roots is equal to the absolute term.*

E.g. the sum of the roots of $4x^2 + 5x - 3 = 0$ is $-\frac{5}{4}$, their product is $-\frac{3}{4}$.

354. Formation of equations. If r_1 and r_2 denote the roots of the quadratic equation $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, the equation may be written:

$$x^2 - (r_1 + r_2)x + r_1r_2 = 0. \quad (1)$$

Or factoring, $(x - r_1)(x - r_2) = 0. \quad (2)$

To form an equation whose roots are given we may use either (1) or (2).

Ex. 1. Form the equation whose roots are 2 and -3 .

According to (2), $(x - 2)(x + 3) = 0.$

Or $x^2 + x - 6 = 0.$

Ex. 2. Form the equation whose roots are $-\frac{5}{2}$ and $-\frac{3}{2}$.

The sum of the roots $= -4.$

The product of the roots $= +\frac{15}{4}.$

Hence, according to (1), $x^2 + 4x + \frac{15}{4} = 0.$

Multiplying by 4, $4x^2 + 16x + 15 = 0.$

Ex. 3. Form the equation whose roots are $2 + \sqrt{2}$ and $2 - \sqrt{2}$.

The sum of the roots $= 4.$

The product of the roots $= 2.$

Hence the equation is $x^2 - 4x + 2 = 0.$

EXERCISE 132

In each of the following equations determine by inspection the sum and the product of the roots:

1. $x^2 - 7x + 6 = 0.$

4. $5x^2 + 5x + 1 = 0.$

2. $x^2 + 8x - 2 = 0.$

5. $x^2 - (a + b)x + ab = 0.$

3. $3x^2 + 5x + 3 = 0.$

6. $7x^2 - x + 1 = 0.$

Form the equations whose roots are:

- | | |
|--------------------------------|--|
| 7. 1, 2. | 14. $-\frac{1}{2}, -\frac{3}{2}$. |
| 8. $-5, -6$. | 15. $3+\sqrt{2}, 3-\sqrt{2}$. |
| 9. $-1, +6$. | 16. $a+\sqrt{b}, a-\sqrt{b}$. |
| 10. $-a, +a$. | 17. $\frac{1+\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}$. |
| 11. $\frac{1}{3}, 3$. | 18. $\frac{a+\sqrt{-b}}{4}, \frac{a-\sqrt{-b}}{4}$. |
| 12. $4+\sqrt{3}, 4-\sqrt{3}$. | |
| 13. ab, a . | |

Solve the following equations, and check the answers by forming the sum and the product of the roots:

- | | |
|--------------------------|--------------------------|
| 19. $x^2 - 4x + 1 = 0$. | 21. $x^2 - 6x + 4 = 0$. |
| 20. $x^2 - 6x + 6 = 0$. | 22. $x^2 + x + 1 = 0$. |

23. Without solving find the sum of the squares of the roots of the equation $ax^2 + bx + c = 0$.

24. Without solving find the difference of the roots of the equation $ax^2 + bx + c = 0$.

FACTORING OF QUADRATIC EXPRESSIONS

355. Let r_1 and r_2 denote the roots of the equations

$$ax^2 + bx + c = 0.$$

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a(x^2 - [r_1 + r_2]x + r_1r_2). \quad (\S\ 354.) \end{aligned}$$

Or factoring, $ax^2 + bx + c = a(x - r_1)(x - r_2)$.

356. Hence any quadratic expression can be factored. The factors, however, are rational only if the roots of the equation obtained by making the expression equal to zero are rational.

Ex. 1. Determine whether $6x^2 + 9x + 2$ has rational factors.

The discriminant $= 9^2 - 4 \cdot 6 \cdot 2 = 33$.

Hence the roots are irrational, and the expression has no rational factors.

Ex. 2. Factor $3x^2 - 19x - 14$.

Solving the equation $3x^2 - 19x - 14 = 0$ by the formula,

$$x = \frac{19 \pm \sqrt{19^2 + 4 \cdot 3 \cdot 14}}{6}.$$

Or
$$x = \frac{19 \pm 23}{6} = 7, \text{ or } -\frac{2}{3}.$$

Hence
$$3x^2 - 19x - 14 = 3(x + \frac{2}{3})(x - 7) \\ = (3x + 2)(x - 7).$$

Ex. 3. Factor $2x^2 - 2x + 1$.

Solving the equation by means of the formula, we find the roots

$$\frac{1 \pm \sqrt{-1}}{2}.$$

Hence
$$2x^2 - 2x + 1 = 2\left(x - \frac{1 + \sqrt{-1}}{2}\right)\left(x - \frac{1 - \sqrt{-1}}{2}\right) \\ = 2 \cdot \frac{2x - 1 - \sqrt{-1}}{2} \cdot \frac{2x - 1 + \sqrt{-1}}{2} \\ = \frac{1}{2}(2x - 1 - \sqrt{-1})(2x - 1 + \sqrt{-1}).$$

Ex. 4. Factor $x^2 + xy - 2xy^2 - 2y^2 + 8y^3 - 8y^4$.

The expression is quadratic in respect to x , hence we solve the equation

$$x^2 + x(y - 2y^2) - 2y^2 + 8y^3 - 8y^4 = 0.$$

By formula,
$$x = -2y + 4y^2, \text{ or } y - 2y^2.$$

Hence
$$x^2 + xy - 2xy^2 - 2y^2 + 8y^3 - 8y^4 \\ = (x + 2y - 4y^2)(x - y + 2y^2).$$

NOTE. A quadratic equation cannot have three roots. For if we write the equation $ax^2 + bx + c = 0$ in the form $a(x - r_1)(x - r_2) = 0$, no other value r_3 , not equal to either r_1 or r_2 , can satisfy the equation, as $a(r_3 - r_1)(r_3 - r_2)$ cannot equal zero.

EXERCISE 133

Determine whether the following expressions have rational factors:

- | | |
|--------------------------|-----------------------------------|
| 1. $3x^2 - 7x + 12$. | 6. $9m^2 - 6mn - 8n^2$. |
| 2. $6x^2 + 20x - 65$. | 7. $16x^2 + 101x - 125$. |
| 3. $72x^2 - 145x + 72$. | 8. $15x^2 - 19xy - 56y^2$. |
| 4. $x^2 - 82x + 1572$. | 9. $4x^2 - 20bx + 25b^2 - 9a^2$. |
| 5. $m^2 - mn - 2n^2$. | 10. $(k+1)x^2 + 3kx + k - 1$. |

Resolve into factors:

- | | |
|---|-----------------------------|
| 11. $x^2 - 60x + 899$. | 17. $300x^2 + 811x - 630$. |
| 12. $25x^2 - 100x + 96$. | 18. $x^2 + 214x + 11,448$. |
| 13. $x^2 - 90x + 2009$. | 19. $x^2 + x + 1$. |
| 14. $50x^2 + 315x - 729$. | 20. $x^2 + 3x - 3$. |
| 15. $x^2 - 64x + 1015$. | 21. $2x^2 + x - 2$. |
| 16. $40x^2 - 53xy + 6y^2$. | 22. $x^2 + 1$. |
| 23. $ax^2 + (a^2 + 1)x + a$. | |
| 24. $3x^2 - 4xy + 8xz - 4y^2 + 8yz - 3z^2$. | |
| 25. $p^2 + 7pq + 2q^2$. | |
| 26. $8x^2 - 16xy - 8xz + 6y^2 - 8yz - 30z^2$. | |
| 27. $12p^2 - 18pq + 28pr - 12q^2 + 19qr - 5r^2$. | |
| 28. $x^2 - 3x - 3$. | |

CHAPTER XXI

PROGRESSIONS

357. A **series** is a succession of numbers formed according to some fixed law.

The **terms** of a series are its successive numbers.

ARITHMETIC PROGRESSION

358. An **arithmetic progression** (A.P.) is a series, each term of which, except the first, is derived from the preceding by the addition of a constant number.

The **common difference** is the number which added to each term produces the next term.

Thus each of the following series is an A. P. :

$$3, 7, 11, 15, 19, \dots$$

$$17, 10, 3, -4, -11, \dots$$

$$a, a + d, a + 2d, a + 3d, \dots$$

The common differences are respectively 4, -7 , and d .

The first is an ascending, the second a descending, progression.

359. To find the n th term l of an A. P., the first term a and the common difference d being given.

The progression is $a, a + d, a + 2d, a + 3d$.

Since d is added to each term to obtain the next one,

$2d$ must be added to a , to produce the 3d term,

$3d$ must be added to a , to produce the 4th term,

$(n - 1)d$ must be added to a , to produce the n th term.

Hence

$$l = a + (n - 1)d. \quad (\text{I})$$

Thus the 12th term of the series 9, 12, 15 is $9 + 11 \cdot 3$ or 42.

360. To find the sum s of the first n terms of an A.P., the first term a , the last term l , and the common difference d being given.

$$s = a + (a + d) + (a + 2d) \cdots (l - d) + l.$$

Reversing the order,

$$s = l + (l - d) + (l - 2d) \cdots (a + d) + a.$$

Adding, $2s = (a + l) + (a + l) + (a + l) \cdots (a + l) + (a + l).$

Or $2s = n(a + l).$

Hence $s = \frac{n}{2}(a + l).$ (II)

Thus to find the sum of the first 50 odd numbers, 1, 3, 5 ... we have from (I),

$$l = 1 + 49 \cdot 2 = 99.$$

Hence $s = \frac{50}{2}(1 + 99) = 2500.$

361. In most problems relating to A.P., five quantities are involved; hence if any three of them are given, the other two may be found by the solution of the simultaneous equations:

$$\begin{cases} l = a + (n - 1)d. & \text{(I)} \end{cases}$$

$$\begin{cases} s = \frac{n}{2}(a + l). & \text{(II)} \end{cases}$$

NOTE. It is possible to find general formulæ expressing any two quantities in terms of any three others. The formulæ, however, have little value, since all examples can be solved without them.

Ex. 1. The first term of an A.P. is 12, the last term 144, and the sum of all terms 1014. Find the series.

$$s = 1014, a = 12, l = 144.$$

Substituting in (I) and (II),

$$144 = 12 + (n - 1)d. \quad (1)$$

$$1014 = \frac{n}{2}(12 + 144). \quad (2)$$

From (2), $78n = 1014$, or $n = 13.$

Substituting in (1), $144 = 12 + 12 \cdot d.$

Hence $d = 11.$

The series is, 12, 23, 34, 45, 56, 67, 78, 89, 100, 111, 122, 133, 144.

Ex. 2. Find n , if $s = 204$, $d = 6$, $l = 49$.

$$\text{Substituting,} \quad 49 = a + (n - 1) \cdot 6. \quad (1)$$

$$204 = \frac{n}{2} (a + 49). \quad (2)$$

$$\text{From (1),} \quad a = 49 - (n - 1) \cdot 6.$$

$$\text{Substituting in (2),} \quad 204 = \frac{n}{2} (98 - \overline{n - 1} \cdot 6).$$

$$408 = n (104 - 6 n).$$

$$6 n^2 - 104 n + 408 = 0.$$

$$3 n^2 - 52 n + 204 = 0.$$

$$\text{Solving,} \quad n = 6, \text{ or } 11\frac{1}{3}.$$

But evidently n cannot be fractional, hence $n = 6$.

EXERCISE 134

1. Find the 11th term of the series 11, 22, 33, ...
2. Find the 18th term of the series 94, 87, 80, ...
3. Find the 7th term of the series 12, $15\frac{1}{2}$, 19.
4. Find the 15th term of the series 8, $10\frac{1}{3}$, $12\frac{2}{3}$.
5. Find the 11th term of the series -1 , $-3\frac{1}{2}$, -6 .
6. Find the 12th term of the series -7 , -1 , $+5$.
7. Find the 13th term of the series -8.5 , -10.9 , -13.3 .

Find the last term and the sum of the following series :

8. 4, 7, 10, ..., to 11 terms.
9. -2 , -1 , 0, ..., to 14 terms.
10. 4, -1 , -6 , ..., to 13 terms.
11. $\frac{1}{5}$, $\frac{4}{5}$, $\frac{7}{5}$, ..., to 10 terms.

Find the sums of the following series :

12. $x - 5$, $x - 4$, $x - 3$, ..., to 11 terms.
13. $-1 \cdot 4$, $-1 \cdot 3$, $-1 \cdot 2$, ..., to 11 terms.

14. $1\frac{1}{2}, 3\frac{1}{4}, 5, \dots$, to 5 terms.
15. $2x + 3y, 3x + 2y, 4x + y, \dots$, to 7 terms.
16. Find the sum of the first 25 integral numbers.
17. Prove that the sum of the first n integral numbers is $\frac{n(n+1)}{2}$.
18. Find the sum of the first 49 odd numbers.
19. Find the sum of the first n odd numbers.
20. Find the sum of the first 40 even numbers.
21. The first term of an A.P. is 13, the last term 88, and the common difference is 5. Find the number of terms.
22. The first term of an A.P. is 62, the last term is 7, and the common difference is -5 . Find the number of terms.
23. The last term of an A.P. of 22 terms is 5, and the common difference is $\frac{1}{3}$. Find the first term and the sum.
24. Given $a=4, l=25, n=8$. Find d .
25. Given $a=7, d=4, n=43$. Find l and s .
26. Given $a=-3, d=2, n=8$. Find l and s .
27. Given $a=4, n=12, l=26$. Find d and s .
28. Given $a=5, n=4, l=-2$. Find d and s .
29. Given $a=1, n=35, s=1225$. Find d and l .
30. Given $n=12, d=4, s=99$. Find a and l .
31. Given $d=1\frac{1}{2}, n=33, l=77$. Find a and s .
32. Given $a=21, l=-59, s=-323$. Find d and n .
33. Given $l=23, s=58, n=29$. Find a and d .
34. Given $a=-7, d=3, s=430$. Find n and l .
35. Given $l=97, d=3, s=1612$. Find a and n .
36. Find d in terms of a, n , and l .
37. Find n in terms of a, l , and s .

362. When three numbers are in A. P. the second one is called the **arithmetic mean** between the other two.

Thus x is the arithmetic mean between a and b , if a , x , and b form an A. P., or if

$$x - a = b - x.$$

Solving,

$$x = \frac{a + b}{2}.$$

I.e. the arithmetical mean between two numbers is equal to half their sum.

Ex. 1. Insert 5 arithmetic means between 14 and 62.

The number of terms is evidently 7.

Hence $n = 7$, $a = 14$, $l = 62$.

Substituting in (I), $62 = 14 + 6 \cdot d$.

Solving, $d = 8$.

Therefore the series is 14, 22, 30, 38, 46, 54, 62.

Or the means are 22, 30, 38, 46, 54.

Ex. 2. The eleventh term of an A. P. is 39, the nineteenth term is 67. Find the series.

Using formula (I) twice, $39 = a + 10 \cdot d$. (1)

$67 = a + 18d$. (2)

Subtracting (1) from (2), $28 = 8d$.

Whence $d = \frac{7}{2}$.

Substituting in (1), $39 = a + 35$.

Therefore $a = 4$.

Or the series is 4, $7\frac{1}{2}$, 11, $14\frac{1}{2}$, ...

EXERCISE 135

1. Insert 6 arithmetic means between 25 and 4.
2. Insert 4 arithmetic means between 17 and -13 .
3. Insert 5 arithmetic means between -1 and -7 .
4. Insert 2 arithmetic means between -9 and 12.
5. Insert 9 arithmetic means between $2\frac{1}{2}$ and $27\frac{1}{2}$.

Find the arithmetic means between :

6. a and $5a$.

8. $\frac{x^2 - y^2}{x}$ and $\frac{x^2 + y^2}{x}$.

7. $\frac{1}{a}$ and $\frac{1}{b}$.

9. $\frac{a - b}{2}$ and $\frac{3a + b}{2}$.

Find the series in which :

10. The 5th term is 19, and the 8th term is 31.

11. The 9th term is 12, and the 17th term is 60.

12. The 4th term is -1 , and the 9th term is -2 .

13. The 12th term of an A. P. is 14, and the 20th term is -6 . Find the 28th term.

14. The sum of three numbers in A. P. is 18, and their product is 192. Find the numbers.

HINT. Let $x - y$, x , $x + y$, represent the numbers.

15. The sum of three numbers in A. P. is 9, and the sum of their squares is 29. What are the numbers ?

16. The sum of five numbers in A. P. is 25, and the sum of their squares is 135. Find the numbers.

17. The sum of four numbers in A. P. is 20, and their product is 384. Find the numbers.

HINT. Let $x - 3y$, $x - y$, $x + y$, $x + 3y$ represent the A. P.

18. How many times does a common clock strike in 12 hours?

19. A bookkeeper receives every year an increase of \$ 100. How many dollars did he receive in 15 years if he received \$ 1200 during the first year ?

20. For boring a well 200 yards deep a contractor receives \$1.00 for the first yard, and for each yard thereafter 1¢ more than for the preceding one. How much does he receive all together ?

21. A man saved each month \$1.00 more than in the preceding one, and all his savings in 10 years amounted to \$8460. How much did he save the first month? How much the last?

22. \$2000 is divided among eight persons so that each person receives \$10 more than the preceding one. How much does each receive?

23. A man accepts a position at a salary of \$1200 for the first year, and is to receive every year \$50 increase of salary. In how many years will he have received \$45,000?

24. If a body falls 5 meters during the first second, 3 times as far during the next second, 5 times as far during the next second, etc., how far will it fall during the 7th second? during the n th second?

25. How far will a body fall in 7 seconds? in t seconds?

26. A man pays off a debt of \$24,000 by monthly payments of \$100, and the first payment is made at the end of the first month. If the yearly rate of interest is 6%, what is the total amount of interest paid until the debt is cleared off?

27. A man purchases a \$500 piano by paying monthly installments of \$10 and interest on the debt. If the yearly rate of interest is 6%, what is the total amount of interest?

GEOMETRIC PROGRESSION

363. A **geometric progression** (G. P.) is a series each term of which, except the first, is derived from the preceding one by multiplying it by a constant number, called the *ratio*.

E.g. 4, 12, 36, 108, ...

4, -2 , $+1$, $-\frac{1}{2}$, ...

a , ar , ar^2 , ar^3 , ...

The ratios are respectively 3, $-\frac{1}{2}$, and r .

364. To find the n th term / of a G. P.; the first term a and the ratios r being given.

The progression is a , ar , ar^2 , ...

To obtain the n th term a must evidently be multiplied by r^{n-1} .

Hence
$$l = ar^{n-1}. \quad (I)$$

Thus the 5th term of the series 16, 24, 36, ..., is $16(\frac{3}{2})^4$ or 81.

365. To find the sum s of the first n terms of a G. P., the first term a and the ratio r being given.

$$s = a + ar + ar^2 \dots ar^{n-1}. \quad (1)$$

Multiplying by r , $rs = ar + ar^2 \dots + ar^n. \quad (2)$

Subtracting (1) from (2),

$$s(r - 1) = ar^n - a.$$

Therefore
$$s = \frac{ar^n - a}{r - 1}. \quad (II)$$

Thus the sum of the first 6 terms of the series 16, 24, 36,

$$s = \frac{16 \left[\left(\frac{3}{2} \right)^6 - 1 \right]}{\frac{3}{2} - 1} = 32 \left(\frac{729}{64} - 1 \right) = 332\frac{1}{2}.$$

NOTE. If n is less than unity, it is convenient to write formula (II) in the following form :

$$s = \frac{a - ar^n}{1 - r}. \quad (III)$$

366. In most problems relating to G. P. five quantities are involved; hence, if any three of them are given, the other two may be found by the solution of the simultaneous equations :

$$\begin{cases} l = ar^{n-1}, \\ s = \frac{ar^n - a}{r - 1}. \end{cases} \quad \begin{matrix} (I) \\ (II) \end{matrix}$$

Ex. 1. To insert 5 geometric means between 9 and 576.

Evidently the total number of terms is 5 + 2 or 7.

Hence $n = 7$, $a = 9$, $l = 576$.

Substituting in I, $576 = 9r^6$.

$$r^6 = 64.$$

$$r = \pm 2.$$

Hence the series is 9, 18, 36, 72, 144, 288, 576.

or 9, -18, 36, -72, 144, -288, 576.

And the required means are $\pm 18, 3, \pm 762, 144, \pm 288$.

EXERCISE 136

1. Find the 8th term of the series 5, 10, 20, ...
2. Find the 8th term of the series 2, 6, 18, ...
3. Find the 7th term of the series 3, -6, +12, ...
4. Find the 6th term of the series 4, -6, +9, ...
5. Find the 10th term of the series 81, 27, 9, ...
6. Find the 7th term of the series $-4, +\frac{8}{3}, -\frac{16}{9}, \dots$

Find the sum of the following series :

7. 2, 6, 18, ... to 6 terms.
8. 2, -4, 8, ... to 7 terms.
9. $1, \frac{1}{2}, \frac{1}{4}, \dots$ to 8 terms.
10. 27, 18, 12, ... to 6 terms.
11. 48, 36, 27, ... to 5 terms.
12. 729, -243, 81, ... to 5 terms.
13. a^7, a^8, a^9, \dots to 8 terms.
14. $a^9 + a^8b + a^7b^2, \dots$ to 7 terms.
15. $a^9 - a^8b + a^7b^2, \dots$ to 7 terms.
16. Find the geometric mean between 2 and 50.
17. Prove that the geometric mean of two numbers is equal to the mean proportional between the numbers.
18. Find the geometric mean between $a^2 - b^2$ and $\frac{a+b}{a-b}$.
19. Insert 3 geometric means between 4 and 324.
20. Insert 3 geometric means between 3 and 48.
21. Given $r=4, n=3, l=80$, find a and s .
22. Given $r=3, n=3, l=18$, find a and s .
23. Given $r=4, n=3, s=105$, find a and l .
24. Given $r=5, n=4, s=780$, find a and l .

25. Given $a=2$, $r=4$, $l=32$, find n and s .
26. Given $a=5$, $r=4$, $l=80$, find n and s .
27. Given $a=4$, $n=3$, $l=64$, find r and s .
28. Given $a=5$, $n=3$, $l=125$, find r and s .
29. Given $a=15$, $r=3$, $s=600$, find n and l .
30. Given $a=15$, $r=4$, $s=5115$, find n and l .
31. Find s in terms of a , r , and l .
32. Find a in terms of r , n , and l .
33. Find a in terms of r , n , and s .
34. Find r in terms of a , n , and l .
35. Find the sum of the series $\sqrt{3}$, 3 , $3\sqrt{3}$, ... to 5 terms.
36. Find the sum of the series 1, 2, 4, 8, to n terms.
37. The fourth term of a G. P. is 135, the seventh term 3645. Find the series.
38. The sum of the third and fifth terms of a G. P. is 90, and the sum of the sixth and eighth terms is 2430. Find the series.
39. The population of a city is 100,000, and it increases 50% every 4 years. What will the population be in 20 years?
40. A sum of money invested at 6% compound interest doubles itself in 12 years. What will \$1.00 invested at 6% compound interest amount to in 240 years?

INFINITE GEOMETRIC PROGRESSION

367. If the value of r of a G. P. is less than unity, the value of r^n decreases, if n increases. The formula for the sum may be written

$$s = \frac{a - ar^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

By taking n sufficiently large, r^n , and hence $\frac{ar^n}{1 - r}$, may be made less than any assignable number.

Consequently, by taking a sufficiently large number of terms, s can be made to differ from $\frac{a}{1-r}$ by less than any given number, however small. This is usually expressed by the formula

$$s_{\infty} = \frac{a}{1-r},$$

where s_{∞} denotes the "sum to infinity."

Ex. 1. Find the sum to infinity of the series $1, -\frac{1}{3}, \frac{1}{9}, \dots$

$$a = 1, r = -\frac{1}{3}.$$

Therefore

$$s_{\infty} = \frac{1}{1 + \frac{1}{3}} = \frac{3}{4}.$$

Ex. 2. Find the value of $.372727 \dots$.

$$.372727 \dots = .3 + .072 + .00072 + \dots$$

The terms after the first form an infinite G. P.

$$a = .072, r = .01.$$

Hence

$$s_{\infty} = \frac{.072}{1 - .01} = \frac{.072}{.99} = \frac{72}{990} = \frac{4}{55}.$$

$$\text{Therefore } .37272 \dots = \frac{3}{10} + \frac{4}{55} = \frac{41}{110}.$$

EXERCISE 137

Find s_{∞} for each of the following series:

$$1. \ 1, \frac{1}{2}, \frac{1}{4} \dots \quad 3. \ 2, -\frac{2}{5}, +\frac{2}{25} \dots \quad 5. \ 3, -\frac{3}{4}, +\frac{3}{16} \dots$$

$$2. \ 1, \frac{1}{3}, \frac{1}{9} \dots \quad 4. \ 4, 1, \frac{1}{4} \dots$$

Find the value of:

$$6. \ .777 \dots \quad 8. \ .126126 \dots \quad 10. \ .42111 \dots$$

$$7. \ .545454 \dots \quad 9. \ .42727 \dots \quad 11. \ 3, \sqrt{3}, 1 \dots$$

12. The sum of an infinite G. P. is 6, and the first term is

4. Find the series.

13. A ball is thrown vertically upwards to a height of 16 feet. After striking the ground it rebounds to three fourths the height it dropped from, and so on for each successive rebound. What is the entire distance traveled by the ball until it comes to rest?

14. Under the conditions given in Ex. 13, the time the ball needs for the first ascension and fall is 2 seconds, and the time between any two successive rebounds is equal to the preceding similar period multiplied by $\frac{1}{2}\sqrt{3}$. In what time does the ball come to rest?

CHAPTER XXII

BINOMIAL THEOREM

PROOF BY MATHEMATICAL INDUCTION

368. The following example explains the **demonstration by mathematical induction**, a method frequently used in algebra.

To prove that

$$1^3 + 2^3 + 3^3 \dots n^3 = \frac{n^2(n+1)^2}{4}.$$

Assuming that the proposition was correct for some number k , we have

$$1^3 + 2^3 + 3^3 \dots k^3 = \frac{k^2(k+1)^2}{4}. \quad (1)$$

Adding $(k+1)^3$ to each member,

$$\begin{aligned} 1^3 + 2^3 + 3^3 \dots (k+1)^3 &= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\ &= \frac{(k+1)^2(k^2 + 4[k+1])}{4}. \end{aligned}$$

$$\text{Or} \quad 1^3 + 2^3 + 3^3 \dots (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}. \quad (2)$$

But (2) expresses that the proposition is true for $(k+1)$. Hence, if the proposition is true for any particular number, it must be true for the next higher number.

Evidently the proposition is true for $k=1$, hence it is true for $k=2$, and therefore again it is true for $k=3$, and so on for higher and higher numbers.

As this mode of increasing k can never reach an end, the proposition is generally true.

369. The method of proving a proposition by mathematical induction consists therefore of two parts.

(1) *Assume the proposition to be true for some number, and demonstrate that it then must be true for the next higher one.*

(2) *Prove that the proposition is true for some particular number.*

EXERCISE 138

Prove the following identities by mathematical induction :

$$1. \quad 1 + 2 + 3 + \cdots + n = \frac{1}{2} n(n+1).$$

$$2. \quad 2 + 4 + 6 + \cdots + 2n = n(n+1).$$

$$3. \quad 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1).$$

$$4. \quad \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

$$5. \quad \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}.$$

$$6. \quad \frac{1}{2 \cdot 3 \cdot 4} + \frac{2}{3 \cdot 4 \cdot 5} + \frac{3}{4 \cdot 5 \cdot 6} + \cdots \\ + \frac{n}{(n+1)(n+2)(n+3)} = \frac{n(n+1)}{4(n+2)(n+3)}.$$

$$7. \quad 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{1}{3} n(n+1)(n+2).$$

$$8. \quad 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots + n(n+1)(n+2) \\ = \frac{1}{4} n(n+1)(n+2)(n+3).$$

$$9. \quad \frac{1}{3 \cdot 4 \cdot 5} + \frac{2}{4 \cdot 5 \cdot 6} + \frac{3}{5 \cdot 6 \cdot 7} + \cdots \\ + \frac{n}{(n+2)(n+3)(n+4)} = \frac{n(n+1)}{6(n+3)(n+4)}.$$

$$10. \quad 1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \cdots + n \cdot 2^{n-1} = (n-1)2^n + 1.$$

$$11. \quad 1 + 2 \cdot 3 + 3 \cdot 3^2 + 4 \cdot 3^3 + \cdots + n \cdot 3^{n-1} = \frac{(2n-1)3^n + 1}{4}.$$

BINOMIAL THEOREM FOR INTEGRAL EXPONENTS

370. It has been shown in § 220 that

$$(a+b)^5 = a^5 + 5a^4b + \frac{5 \cdot 4}{1 \cdot 2} a^3b^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} a^2b^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} ab^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} b^5.$$

The method of induction furnishes a convenient means for proving this proposition for any positive integral exponents, *i.e.* to prove

$$(a+b)^n = a^n + n \cdot a^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 \dots b^n. \quad (1)$$

Assuming that the proposition is true for a number k , we have

$$(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{1 \cdot 2} a^{k-2}b^2 + \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3} a^{k-3}b^3 \dots$$

Multiplying both members by $(a+b)$, and collecting similar terms in the right member,

$$(a+b)^{k+1} = a^{k+1} + (k+1)a^k b + \left\{ \frac{k(k-1)}{2} + k \right\} a^{k-1}b^2 + \left\{ \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3} + \frac{k(k-1)}{1 \cdot 2} \right\} a^{k-2}b^3 \dots$$

$$\text{Or } (a+b)^{k+1} = a^{k+1} + (k+1)a^k b + \frac{(k+1)k}{1 \cdot 2} a^{k-1}b^2 + \frac{(k+1)k(k-1)}{1 \cdot 2 \cdot 3} a^{k-2}b^3 \dots \quad (2)$$

But (2) expresses that the proposition is true for the exponent $(k+1)$, since every k of (1) has been replaced in (2) by a $(k+1)$.

Therefore, if the proposition is true for any exponent, it must be true for the next higher one, and since it is true for $k=2$, it is generally true. (§ 369.)

371. The product $1 \cdot 2 \cdot 3 \cdots n$ is called **factorial n** , and it is frequently represented by the symbols $\lfloor n$ or $n!$

Thus $\lfloor 3 = 1 \cdot 2 \cdot 3 = 6$; $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

372. Using this symbol, the binomial theorem may be written,

$$(a + b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{\lfloor 2} a^{n-2}b^2 + \dots$$

$$\frac{n(n-1) \cdots (n-r+1)}{\lfloor r} a^{n-r}b^r \cdots + b^n.$$

The student should note that every coefficient of the expansion has the same number of factors in numerator and denominator.

373. The general term of expansion, *i.e.* the

$$(r+1)\text{th term} = \frac{n(n-1) \cdots (n-r+1)}{\lfloor r} a^{n-r}b^r.$$

Ex. 1. Expand $(2a^2 - \sqrt{b})^5$.

$$\begin{aligned} (2a^2 - \sqrt{b})^5 &= (2a^2 - b^{\frac{1}{2}})^5 \\ &= (2a^2)^5 - 5 \cdot (2a^2)^4 b^{\frac{1}{2}} + \frac{5 \cdot 4}{1 \cdot 2} (2a^2)^3 (b^{\frac{1}{2}})^2 \\ &\quad - \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} (2a^2)^2 (b^{\frac{1}{2}})^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3 \cdot 4} (2a^2) (b^{\frac{1}{2}})^4 - (b^{\frac{1}{2}})^5 \\ &= 32a^{10} - 80a^8b^{\frac{1}{2}} + 80a^6b - 40a^4b^{\frac{3}{2}} + 10a^2b^2 - b^{\frac{5}{2}}. \end{aligned}$$

Ex. 2. Find the 7th term in the expansion of $(3a + b^2)^8$.

Since $r+1=7$, $r=6$, and $n=8$.

$$\text{Hence the 7th term} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} (3a)^2 (b^2)^6 = 252 a^2 b^{12}.$$

Ex. 3. Find the middle term of $(x-2y)^{10}$.

Since the expansion has 11 terms, the 6th term is the middle term, or

$$r+1=6.$$

Therefore $r=5$, $n=10$, $a=x$, $b=-2y$.

$$\begin{aligned} \text{Hence the middle term} &= \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} x^5 (-2y)^5 \\ &= -8064 x^5 y^5. \end{aligned}$$

EXERCISE 139

Expand the following :

- | | | |
|----------------|---|--|
| 1. $(x+y)^5$. | 5. $\left(\frac{a}{b} - \frac{b}{a}\right)^5$. | 7. $(x^2 - 3y^3)^4$. |
| 2. $(x+3)^4$. | | 8. $\left(ax - \frac{1}{x^2}\right)^5$. |
| 3. $(m-n)^6$. | 6. $\left(x + \frac{1}{x}\right)^4$. | |
| 4. $(1+x)^8$. | | |

Simplify :

9. $(\sqrt{a} + \sqrt{b})^4 + (\sqrt{a} - \sqrt{b})^4$. 11. $(1 + \sqrt{x})^5 - (1 - \sqrt{x})^5$.
10. $(1 + \sqrt{n})^6 + (1 - \sqrt{n})^6$. 12. $(\sqrt{2} + x)^6 + (\sqrt{2} - x)^6$.
13. Find the 4th term of $(a+b)^9$.
14. Find the 9th term of $(a-b)^{10}$.
15. Find the 3d term of $(1+x)^7$.
16. Find the 4th term of $(a+3b)^7$.
17. Find the 5th term of $(a^2 - 3b^2)^8$.
18. Find the 7th term of $\left(\sqrt{a} - \frac{1}{a}\right)^{10}$.
19. Find the 10th term of $(1+x)^{12}$.
20. Find the 5th term of $\left(x - \frac{1}{x^2}\right)^9$.
21. Find the 20th term of $(1-x^{-1})^{22}$.
22. Find the middle term of $(a+b)^6$.
23. Find the middle term of $(m-n)^8$.
24. Find the middle term of $\left(x - \frac{1}{x}\right)^{10}$.
25. Find the middle term of $(a^2 - 5ab^2)^6$.
26. Find the middle term of $\left(\sqrt[3]{m} - \frac{1}{m^2}\right)^4$.
27. Find the term independent of x in $\left(x - \frac{1}{x}\right)^{10}$.
28. Find the term independent of x in $\left(x^2 - \frac{1}{x}\right)^6$.

374. The coefficients of the expansion of $(a+b)^n$, called the **binomial coefficients**, are represented by several symbols, *e.g.* nC_r , ${}_nC_r$, C_r^n , and $\binom{n}{r}$.

Using the first symbol,

$${}^5C_2 = \frac{5 \cdot 4}{1 \cdot 2}.$$

$${}^nC_2 = \frac{n(n-1)}{1 \cdot 2}.$$

$${}^nC_r = \frac{n(n-1) \cdots (n-r+1)}{\underline{r}}.$$

375. The binomial theorem may then be written

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 \cdots + {}^nC_r a^{n-r}b^r \cdots + b^n.$$

Making $a=1$ and $b=x$,

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_r x^r \cdots + x^n.$$

376. The $(r+1)$ term may be written

$${}^nC_r a^{n-r} b^r.$$

377. If we multiply the numerator and the denominator of nC_r by $\underline{n-r}$,

$${}^nC_r = \frac{n(n-1) \cdots (n-r+1)(n-r) \cdots 2 \cdot 1}{\underline{r} \underline{n-r}}.$$

$${}^nC_r = \frac{\underline{n}}{\underline{r} \underline{n-r}}.$$

Substituting $n-r$ for r ,

$${}^nC_{n-r} = \frac{\underline{n}}{\underline{n-r} \underline{r}}.$$

That is,

$${}^nC_r = {}^nC_{n-r}.$$

378. Hence the binomial coefficients of any terms equidistant from the end and the beginning are equal.

Thus, ${}^{10}C_8 = {}^{10}C_2 = \frac{10 \cdot 9}{1 \cdot 2} = 45.$

$${}^{100}C_{99} = {}^{100}C_1 = 100.$$

379. By substituting $a=1$ and $b=1$ in the formula for the binomial theorem, we have

$$2^n = 1 + {}^nC_1 + {}^nC_2 \dots {}^nC_n$$

That is, the sum of the coefficients of the expansion of $a+b$ is 2^n .

Similarly, the sum of the coefficients of the expansions of $(a+2b)^n = 3^n$, etc.

380. In higher algebra it is proved that the binomial theorem is true for negative and fractional values of n , provided a is greater than b . The expression, however, gives an infinite number of terms.

$$\begin{aligned} \text{Thus, } (1+x)^{\frac{1}{2}} &= 1 + \frac{1}{2}x + \frac{\frac{1}{2} \cdot -\frac{1}{2}}{2}x^2 + \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2}}{\underline{3}}x^3 \dots \\ &= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 \dots \end{aligned}$$

381. The sum of two successive binomial coefficients is a binomial coefficient of the next higher order.

$${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r.$$

$$\begin{aligned} \text{For } {}^nC_r + {}^nC_{r-1} &= \frac{n(n-1) \dots (n-r+1)}{1 \cdot 2 \dots r} + \frac{n(n-1) \dots (n-r+2)}{1 \cdot 2 \dots (r-1)} \\ &= \frac{n(n-1) \dots (n-r+2)}{1 \cdot 2 \cdot 3 \dots (r-1)} \left\{ \frac{n-r+1}{r} + 1 \right\} \\ &= \frac{n(n-1) \dots (n-r+2)}{1 \cdot 2 \cdot 3 \dots (r-1)} \cdot \frac{(n+1)}{r} \\ &= \frac{(n+1)n \cdot (n-1) \dots (n-r+2)}{1 \cdot 2 \cdot 3 \dots r} = {}^{n+1}C_r. \end{aligned}$$

$$E.g. \quad {}^5C_3 + {}^5C_2 = {}^6C_3, \quad {}^nC_6 + {}^nC_5 = {}^{n+1}C_6.$$

382. The proof for the binomial theorem, as given in § 370, referred only to the first three terms. To prove that it is correct for any term we may use the proposition of the preceding paragraph. Assuming that

$$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 \dots {}^nC_{r-1}x^{r-1} + {}^nC_rx^r \dots {}^nC_nx^n.$$

Multiplying by $(1 + x)$,

$$(1 + x)^{n+1} = \begin{cases} 1 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_{r-1}x^{r-1} + {}^nC_rx^r + \dots + x^n \\ + x + {}^nC_1x^2 + \dots + {}^nC_{r-1}x^r + \dots + x^{n+1}. \end{cases}$$

$$(1 + x)^{n+1} = 1 + {}^{n+1}C_1x + {}^{n+1}C_2x^2 + \dots + {}^{n+1}C_rx^r + \dots + x^{n+1}.$$

Hence it follows by the method used in § 369 that the theorem is correct for all terms.

EXERCISE 140

1. Find ${}^{1000}C_{999}$, ${}^{102}C_{100}$.
2. Find the 98th term of $(a + b)^{100}$.
3. Find the 47th term of $(1 - x)^{50}$.
4. Find the sum of the coefficients in the expansion of $(1 + x)^{10}$.
5. Find the sum of the coefficients in the expansion of $(1 - x)^{10}$.
6. Find the sum of the coefficients in the expansion of $(2a + b)^5$.
7. Find the sum of 6C_1 , 6C_2 , 6C_3 , 6C_4 , 6C_5 , and 6C_6 without finding the value of each coefficient.
8. Find n , if ${}^nC_2 = 28$.
9. Find n , if ${}^nC_3 \div {}^nC_2 = 10$.
10. Expand $(1 + x)^{-1}$ to four terms.
11. Expand $(1 - x)^{-1}$ to four terms.
12. Expand $(1 - x)^{\frac{1}{2}}$ to four terms.
13. Write the term of $\left(1 + \frac{3}{x^2}\right)^{10}$, which contains x^{-6} .
14. Write the term of $\left(x - \frac{1}{x}\right)^9$, which contains x^5 .
15. In the expansion $(a + b)^{10}$, the coefficient of the r th term is equal to the coefficient of the $2r$ th term. Find r .

REVIEW EXERCISE V

Simplify :

✓ 1. $(\frac{1}{3} - 6z^{-2} + 27z^{-4}) \div (\frac{1}{3} + 2z^{-1} + 3z^{-2}).$

✓ 2. $(\sqrt[m]{a^{-mn}} + \sqrt[p]{a^{-pn}})^2.$

3. $\left(n^2x^{2n} - n(n-1)x^{2n-1} + \frac{(n-1)^2}{4}x^{2n-2}\right)^{\frac{1}{2}}.$

✓ 4. $2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} + \sqrt{\frac{3}{5}}.$

✓ 5. $7\sqrt[3]{54} + 3\sqrt[3]{16} + \sqrt[3]{2} - 5\sqrt[3]{128}.$

✓ 6. $(9 + 2\sqrt{10})(9 - 2\sqrt{10}).$

✓ 7. $(2\sqrt{8} + 3\sqrt{5} - 7\sqrt{2})(\sqrt{72} - 5\sqrt{20} - 2\sqrt{2}).$

8. $(\sqrt{3} + \sqrt{2}) \div (\sqrt{3} - \sqrt{2}).$

9. $1 \div (\sqrt[4]{2} - 1).$

11. $(5 - \sqrt{24})^{\frac{1}{2}}.$

10. $(7 + 4\sqrt{3})^{\frac{1}{2}}.$

12. $(x + 2\sqrt{x-1})^{\frac{1}{2}}.$

✓ 13. $\frac{x-2}{x^2-x+1} - \frac{x+2}{x^2+x+1} + \frac{1}{x-1} - \frac{1}{x+1}.$

✓ 14. $\frac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^2(x-a)(x-c)}{(b-a)(b-c)} + \frac{c^2(x-a)(x-b)}{(c-a)(c-b)}.$

15. $\frac{a^2 + b^2 + c^2 + 2ab + 2ac + 2bc}{a^2 - b^2 - c^2 - 2bc}.$

✓ 16. $\frac{8a - 5x + 8ay - 5xy}{24x + 16ax + 12xy + 8axy} \div \left\{ \frac{2a}{2ax + 3x} - \frac{5}{8a + 12} \right\}$

17. Find the coefficient of $a^{13}b^2$ in $(a+b)^{15}.$

18. Find the coefficient of a^{97} in $(a-1)^{100}.$

19. Find the middle term of the expansion of $(a-b)^{16}.$

20. Find s_{∞} for the series,

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} \dots$$

21. Solve $x = 1 + \frac{1}{1 + \frac{1}{x}}$.

22. Solve $\sqrt{\frac{20}{x^2} + 9} - \sqrt{\frac{20}{x^2} - 9} = 3$.

23. $x^2 = 1 - x$. Find x to four decimal places.

24. Solve $x^2 - \frac{a^2 + b^2}{ab}x + 1 = 0$.

25. Solve $32a^{2m}c^{n-1} + 4a^{m+3}c^{n-1}(ac^3 - 2)x = a^7c^{n+2}x^2$.

26. Find the equation whose roots are a and $\frac{1}{a}$.

27. If the equation $x^2 - 8x + 15 = k$ has equal roots, what is the value of k ?

28. Solve $\begin{cases} x^2 + 3y^2 = \frac{7}{9}, \\ 4x^2 + 2y^2 = 2. \end{cases}$

30. Solve $\begin{cases} x + y = a, \\ x^2 + y^2 = b. \end{cases}$

29. Solve $\begin{cases} x - y = 6, \\ xy = 91. \end{cases}$

31. Solve $\begin{cases} x^2y + xy^2 = 30, \\ \frac{1}{x} + \frac{1}{y} = \frac{5}{6}. \end{cases}$

32. Solve $\begin{cases} (x + y)(x^2 + y^2) = 1400, \\ x^3 + y^3 = 728. \end{cases}$

33. A body projected vertically upward with a velocity of v feet per second reaches after t seconds an elevation $s = vt - \frac{g}{2}t^2$. ($g = 32$ feet.) If $v = 64$, and $s = 48$, find t .

34. Find three numbers such that the product of any two of them divided by the third one gives respectively the quotients a , b , and c .

35. The edges of two hollow cubes differ by 10 centimeters. If a certain quantity of water is poured into the larger cube, there remain 1578 cubic centimeters of space not filled with water. If the second cube contained 142 cubic centimeters more, it would hold the quantity of water. Find the number of cubic centimeters of water.

36. The geometric mean of two numbers is to their arithmetic mean as 3 : 5. Find the ratio of the two numbers.

37. The sum of five numbers in A. P. is 25, and their product is 945. Find the numbers.

38. A man spends \$6 on the first day of January, and his expenses increase 50¢ every day. How much does he spend during the month of January?

39. The population of a town increased during 4 years from 10,000 to 14,641. If the rate of increase was the same every year, find that rate.

40. What is the remainder when $2x^5 + 3x^3 - 2x$ is divided by $x + 2$?

41. For what value of l is $2x^4 - 3x^3 + lx^2 - 9x + 1$ exactly divisible by $x - 3$?

42. Prove that $x^m - 2x^n + 1$ is exactly divisible by $x - 1$.

43. Factor $6x^2 + 59xy + 105y^2$.

✓ 44. Factor $(x^2 - 5x + 2)^2 - 4$.

45. Add to 10 terms $x + y$, $x^2 + 2y$, $x^3 + 3y$, $x^4 + 4y$...

46. Solve $\begin{cases} x^2 + 3xy = 7, \\ xy + 3y^2 = 14. \end{cases}$

47. Solve $\sqrt[3]{a^{5x+3}} = a^{12}$.

48. Solve $(\frac{2}{7})^{3x-7} = (\frac{7}{8})^{7x-3}$.

49. Factor $a^2 - ab + 3a - 2b^2 - 3b + 2$.

50. Solve the equation $x^{\frac{m}{n}} - 3x^{\frac{m}{2n}} + 1 = 0$.

✓ 51. Simplify $2^2 + 10^0 - 4^{\frac{3}{2}} - (\frac{1}{9})^{-\frac{1}{2}} + 0^4 + \sqrt[2]{\frac{1}{16}}$.

52. Solve $\begin{cases} \frac{x}{a} + \frac{y}{b} = 1, \\ \frac{a}{x} + \frac{b}{y} = -1. \end{cases}$

✓ 53. Find the L. C. M. of $x^4 - x^2$, $x^3 - x^2$, $x^2 + 1$.

54. Find the fifth term of

$$\left(\frac{3a^2}{\sqrt[3]{b}} - \frac{\sqrt[3]{a}}{3b^2} \right)^7$$

and reduce it to its simplest form.

✓ 55. Simplify $\frac{1}{a} \div \sqrt{\frac{\sqrt[10]{a^6}}{\sqrt{a} \cdot (\sqrt[3]{a} \cdot \sqrt[5]{a})^2}}$.

CHAPTER XXIII

INEQUALITIES *

383. An **inequality** is a statement that one quantity is greater or less than another.

384. The **signs of inequality**, $>$ and $<$, are respectively read “is greater than” and “is less than.”

The **members** or **sides** of an inequality are the two expressions which are connected by a sign of inequality.

Thus, $a > b$ is read “ a is greater than b .” a is the left, b the right, member.

385. One number a is greater than another number b , if $a - b$ is positive; similarly, $a < b$, if $a - b$ is negative.

$$-5 > -7, \text{ since } -5 - (-7) = +2.$$

386. The symbols, ∇ , \nless , \neq , express respectively “is not greater than,” “is not smaller than,” and “is not equal to.”

387. Two inequalities are of the **same species**, or **subsist in the same sense**, if their signs of inequality are alike.

$a > b$, $c > d$, are of the same species.

$4 > 3$, $-4 < -3$, are of opposite species.

388. *The sense of an inequality is not changed if both members are increased or diminished by the same number.*

Suppose $a > b$, then $a - b$ is positive.

Hence $(a + m) - (b + m)$ is positive.

Therefore $a + m > b + m$.

Similarly $a - m > b - m$.

* This chapter is not required for the examinations of the College Entrance Examination Board.

389. It follows from the preceding paragraph that *a term may be transposed from one member of an inequality to another, by changing its sign.*

390. *The sense of an inequality is not changed if both members are multiplied or divided by the same positive number.*

Let $a > b$, and m be a positive number.

Then $a - b$ is positive.

Hence $m(a - b)$ or $ma - mb$ must be positive; i.e. $ma > mb$.

In a similar manner it can be proved that $\frac{a}{m} > \frac{b}{m}$.

391. *If the signs of all terms of an inequality are changed, the sign of inequality must be reversed.*

Consider the inequality $a - b + c > x - y$.

Transposing all terms, $-x + y > -a + b - c$.

That is, $-a + b - c < -x + y$.

392. *The sense of an inequality is reversed if both members are multiplied or divided by the same negative number.*

This follows from § 390 and § 391.

393. The following principles can easily be demonstrated:

1. *If any number of inequalities of the same species be added, the resulting inequality will be of the same species as the original ones.*

2. *If $a > b$, and $b > c$, then $a > c$.*

3. *The sense of inequalities of the same species is not changed by multiplying their corresponding members, provided all members are positive.*

HINT. If $a > b$, and $x > y$, then $ax > bx$, and $bx > by$. Therefore $ax > by$.

4. *The sense of an inequality is not changed by raising both numbers to the same power, if the signs of the members are not changed by the involution.*

394. An **identical inequality** is one which is true for all values of the letters involved.

Thus, $a + 9 > a$, and $m^2 + n^2 > 0$, are identical inequalities.

395. A **conditional inequality** is one which is true only for certain values of the letters involved.

$x + 1 > 5$ is a conditional inequality. It is true only if $x > 4$.

396. To solve a conditional inequality for a certain letter means to find all values of the letter for which the inequality is true.

Thus, $x - 1 > 2$ is evidently true for all values of x greater than 3. The value 3 is sometimes called the "limit of x " in the inequality $x - 1 > 2$.

397. Conditional inequalities are solved in the same manner as equations. Care, however, should be taken to change the sense of an inequality if the signs of both members are changed by multiplication, division, etc.

Ex. 1. Solve the inequality:

$$\frac{6 - 5x}{12} - \frac{2x + 3}{6} > 1 - \frac{x - 5}{3}$$

Clearing of fractions, $6 - 5x - 4x - 6 > 12 - 4x + 20$.

Transposing, $-5x - 4x + 4x > 12 + 20$.

Uniting, $-5x > 32$.

Dividing by -5 , $x < -6\frac{2}{5}$.

398. Identical inequalities are usually proved by the same method as identical equations (§ 169). The fact that the square of any real number is positive can often be used for such proofs.

Ex. 2. If a and b are unequal, positive numbers, prove that $\frac{a}{b} + \frac{b}{a} > 2$.

The inequality is true,

if $a^2 + b^2 > 2ab$ (clearing of fractions),

or, if $a^2 - 2ab + b^2 > 0$ (transposing),

or, if $(a - b)^2 > 0$.

But the last inequality is true.

Hence $\frac{a}{b} + \frac{b}{a} > 2$.

NOTE. All letters in the present chapter are supposed to represent real numbers.

399. If $a = b$, then $a^2 + b^2 = 2ab$;

if $a \neq b$, then $a^2 + b^2 > 2ab$.

Hence $a^2 + b^2$ is either equal to or greater than $2ab$, a statement that may be written:

$$a^2 + b^2 \geq 2ab, \quad (1)$$

or, $a^2 + b^2 \not< 2ab. \quad (2)$

Many proofs of inequalities can be based upon the identity (1).

Ex. 3. Prove that $a^2 + b^2 + c^2 \geq ab + bc + ac$.

$$a^2 + b^2 \geq 2ab.$$

$$b^2 + c^2 \geq 2bc.$$

$$c^2 + a^2 \geq 2ca.$$

Adding, $2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ac.$

Dividing by 2, $a^2 + b^2 + c^2 \geq ab + bc + ac.$

EXERCISE 141

Solve the following inequalities:

$$1. \frac{3x}{4} - 5 > \frac{2x}{5} + 2. \qquad 3. \frac{x-3}{3} - \frac{x-8}{8} < \frac{5-x}{5}.$$

$$2. \frac{x-2}{2} + \frac{x-4}{4} > \frac{x-3}{3}. \qquad 4. \frac{x}{4} + \frac{x}{5} < -13 - \frac{x}{3}.$$

$$5. \frac{3x+1}{11} - \frac{2x+1}{3} > \frac{2x+1}{5}.$$

$$6. x^2 + (x+1)(x-1) - 2(x-2)(x+3) > 0.$$

$$7. 6 - \frac{2(x-7)}{3} > \frac{x+2}{4} + \frac{x+1}{11}.$$

$$8. x - \frac{7x-3}{4} < \frac{5x+7}{12} - \frac{3x+2}{5}.$$

$$9. (1+6x)^2 - (1+10x)^2 + (2+8x)^2 > 0.$$

$$10. x + \frac{x}{a} > b, \text{ if } a \text{ is positive.}$$

$$11. (a-x)(b+x) > a^2 - x^2, \text{ if } a-b > 0.$$

$$12. \frac{1+x}{1-x} < \frac{a}{b}, \text{ if } a \text{ and } b \text{ are positive.}$$

13. Between what limits must x lie, if

$$\begin{cases} \frac{x-4}{7} + \frac{x+4}{8} < 2, \text{ and} \\ (x+1)(x+2) > x^2 + 5? \end{cases}$$

$$14. \text{ Solve } x^2 - 169 < 0.$$

$$15. \text{ If } \frac{a-b}{b} > \frac{2a+b}{3b}, \text{ and } b \text{ is positive, solve for } a.$$

16. Find the limit of x , if

$$\begin{cases} x+4y=37, \text{ and} \\ 2x+5y>53. \end{cases}$$

If a , b , and c represent unequal, positive numbers, prove that :

17. $a^2 + 3b^2 > 2b(a + b)$.

19. $\frac{a+b}{2} > \frac{ab}{a+b}$.

18. $\frac{a^2 + 5b^2}{4} > \frac{ab}{2} + b^2$.

20. $a^3b + ab^3 > 2a^2b^2$.

21. $(a^4 + b^4)(a^2 + b^2) > (a^3 + b^3)^2$.

22. $a^3 + b^3 > ab(a + b)$. HINT. Divide by $a + b$.

23. $a + b > 2\sqrt{ab}$.

24. $a^3 - b^3 > a^2b - ab^2$.

25. $(a^2 + b^2)(b^2 + c^2)(c^2 + a^2) > 8a^2b^2c^2$.

26. $(a + b)(b + c)(c + a) > 8abc$.

27. Which is the greater, $a^3b + ab^3$ or $a^4 + b^4$?

28. Prove that $a^3 + b^3 + c^3 > 3abc$.

CHAPTER XXIV

VARIABLES AND LIMITS

400. Functions (§ 292) are usually denoted by symbols of the form $f(x)$, $P(x)$, $F(x)$, etc., and are read *f function of x*, *P function of x*, etc.

Thus, $f(x)$ comprises all expressions which involve the letter x , and it may represent in one discussion $3x^2 + 2x + 1$, in another \sqrt{x} , in a third $4^x + 2x^3$, etc.

401. If $f(x)$ is known in any particular discussion, $f(a)$ is formed by substituting a in place of x .

E.g. If $f(x) = 2x^2 + 2x + 3$,
then $f(3) = 2 \cdot 3^2 + 2 \cdot 3 + 3 = 27$,
 $f(a) = 2a^2 + 2a + 3$,
 $f(0) = 0 + 0 + 3 = 3$, etc.

402. If $y = f(x)$, x is called the **independent variable**, and y the **dependent variable**.

403. Similarly, $f(x, y)$ denotes a function of two independent variables. Thus, $x^2 + 3xy + y^2$, or $\sqrt{x} + \sqrt{y}$, or $\frac{x}{y}$, may be represented by the symbol $f(x, y)$.

E.g. If $f(x, y) = xy + x$, $f(2, 3) = 2^3 + 2 = 10$.

EXERCISE 142

1. If $f(x) = x^2 + 3x + 2$, find $f(1)$, $f(0)$, $f(-1)$.
2. If $f(x) = 5x^3 - 2x - 3$, find $f(2)$, $f(a)$, $f(0)$.
3. If $f(x) = 4^x$, find $f(0)$, $f(-1)$, $f(\frac{1}{2})$.

4. If $f(x) = \sqrt[3]{16}$, find $f(2)$, $f(1)$, $f(4)$.
5. If $f(x) = (x - m)(x - n)$, find $f(m)$, $f(n)$, $f(0)$.
6. If $f(x) = x^2 - y^2$ and $F(x) = x - y$, find $\frac{f(x)}{F(x)}$.
7. If $f(x, y) = \sqrt{x^2 + y^2}$, find $f(3, 4)$.
8. Solve the equation $f(x) = f(7)$.
9. Solve the equation $f(x) = f(a + 1)$.
10. Simplify $\frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$.
11. If $f(x) = x^2 + 2x$, find $f(a + h) - f(a)$.
12. If $f(x) = x^3 + 1$, find $f(x + h) - f(x)$.
13. If $f(x) = 2x + 1$, find $\frac{f(a + 1)}{f(a - 1)}$.
14. If $f(x) = a^x$, find $\frac{f(x + h)}{f(x - h)}$.

404. The **limit** of a variable is a constant which the variable approaches indefinitely without becoming equal to it.

The difference between the constant $\frac{2}{3}$ and the variable .6, .66, .666, ... is respectively less than $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, By increasing the number of decimals, this difference can be made smaller than any assignable number, but the decimal can never equal $\frac{2}{3}$.

Hence $\frac{2}{3}$ is the limit of .666 ...

405. The symbol \doteq denotes *approaches as a limit*.

Thus, $x \doteq a$, or $\lim x = a$, denotes x approaches a as a limit.

406. To prove that a variable x approaches a constant a as a limit, it is necessary and sufficient to prove that the absolute value of $a - x$ can become smaller than any assigned constant, however small, without becoming zero.

E.g. Let x equal the sum of n terms of the series

$$1 + \frac{1}{2} + \frac{1}{4} \dots$$

Then

$$x = \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 2 - \frac{1}{2^{n-1}}.$$

Hence

$$2 - x = \frac{1}{2^{n-1}}.$$

By increasing n we can make $\frac{1}{2^{n-1}}$, or $2 - x$, as small as we please.

Thus, to make $2 - x < \frac{1}{1000}$, make $n = 11$,

then
$$2 - x = \frac{1}{2^{n-1}} = \frac{1}{1024}.$$

Similarly, to make $2 - x < \frac{1}{1000000}$, let $n = 21$,

then
$$2 - x = \frac{1}{2^{n-1}} = \frac{1}{1048576}, \text{ etc.}$$

Hence $2 - x$ can be made smaller than any assignable number, but $2 - x$ cannot be made equal to zero.

Therefore x approaches 2 as a limit.

407. If a variable increases so that its absolute value becomes greater than any assignable number, however great, the variable is said to **increase indefinitely** or to become **infinite** (∞).

Thus, $\frac{1}{x}$ increases indefinitely, if $x \doteq 0$.

408. An **infinitesimal** is a variable whose limit is zero.

Thus, $\frac{1}{x}$ becomes infinitesimal, if $x \doteq \infty$.

A **finite** number is a number which is neither *infinite* nor *infinitesimal*.

409. The symbol $[f(x)]_{x \doteq a} \doteq b$, or $\lim [f(x)]_{x \doteq a} = b$, denotes *$f(x)$ approaches b as a limit, if x approaches a as a limit.*

Thus,
$$\lim \left[\frac{1}{x} \right]_{x \doteq 0} = \infty.$$

Or
$$\left[\frac{1}{x} \right]_{x \doteq \infty} \doteq 0.$$

NOTE. The last two statements are frequently written in the abbreviated form

$$\frac{1}{0} = \infty \text{ and } \frac{1}{\infty} = 0. \quad (\S 209 \text{ and } \S 210.)$$

410. *If two variables, x and y , are always equal, and x approaches a as a limit, then y approaches a as a limit.*

Since $a - x = a - y$,

(1) $a - y$ can be made smaller than any assignable number.

(2) $a - y$ cannot equal zero.

Hence $y \doteq a$.

411. *If two variables are always equal, and each approaches a limit, the limits are equal.*

This follows directly from the preceding paragraph.

412. The following relations are frequently used :

$$(1) \lim (x + y) = \lim x + \lim y.$$

$$(2) \lim (xy) = \lim x \times \lim y.$$

$$(3) \lim \left(\frac{x}{y} \right) = \lim x \div \lim y.$$

413. Vanishing fractions. A number of functions assume indeterminate values for certain values of the independent variable, thus :

$$\frac{x-1}{x^2-1} = \frac{0}{0}, \text{ if } x=1.$$

$$\frac{1}{2x} - \frac{1}{x} = \infty - \infty, \text{ if } x=0, \text{ etc.}$$

414. The principal indeterminate form is $\frac{0}{0}$. All others, as $\frac{\infty}{\infty}$, 0^0 , $\infty - \infty$, 0^∞ , ∞^0 , can be reduced to the form $\frac{0}{0}$.

415. Many of these functions, however, approach definite limits, if we let x approach its assigned value as a limit. These limiting values are usually considered the true values of the functions.

Ex. 1. Find the limiting value of $\frac{1-x}{1-x^2}$, if $x \doteq 1$.

Direct substitution produces the value $\frac{0}{0}$, but by reducing the fraction to its lowest terms, we have,

$$\left[\frac{1-x}{1-x^2} \right]_{x \doteq 1} = \left[\frac{1}{1+x} \right]_{x \doteq 1} = \frac{1}{2}.$$

Ex. 2. Find the limiting value of

$$\frac{4x^3 - 2x + 1}{2x^3 - 3x^2 + 4}, \text{ when } x \doteq \infty.$$

Direct substitution produces $\frac{\infty}{\infty}$; hence divide numerator and denominator by x^3 ,

$$\frac{4x^3 - 2x + 1}{2x^3 - 3x^2 + 4} = \frac{4 - \frac{2}{x^2} + \frac{1}{x^3}}{2 - \frac{3}{x} + \frac{4}{x^3}}.$$

If $x \doteq \infty$, $\frac{2}{x^2}$, $\frac{1}{x^3}$, etc., approach 0 as a limit.

Hence the required limit is $\frac{4}{2}$ or 2.

Ex. 3. Find $\lim \left[\frac{x^2}{2x-4} - \frac{3x-4}{x-2} \right]_{x \doteq 2}$.

Direct substitution produces $\infty - \infty$, hence simplify,

$$\frac{x^2}{2x-4} - \frac{3x-4}{x-2} = \frac{x^2 - 6x + 8}{2(x-2)} = \frac{x-4}{2}.$$

If $x \doteq 2$, this value reduces to $-\frac{2}{2}$ or -1 .

Ex. 4. Find $\left[\frac{f(x+h) - f(x)}{h} \right]_{h \doteq 0}$, if $f(x) = x^3$.

$$\begin{aligned} \frac{(x+h)^3 - x^3}{h} &= \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= 3x^2 + 3xh + h^2. \end{aligned}$$

Hence

$$\left[\frac{(x+h)^3 - x^3}{h} \right]_{h \doteq 0} = 3x^2.$$

EXERCISE 143

Find the following limiting values:

1. $\left[\frac{x^2 + 4x + 3}{x^2 + 3x + 2} \right]_{x \rightarrow -1}$.
2. $\left[\frac{3x^2 - 4x + 2}{x^2 + 3} \right]_{x \rightarrow \infty}$.
3. $\left[\frac{x^3 + a^3}{x^2 - a^2} \right]_{x \rightarrow -a}$.
4. $\left[\frac{x^3 - x}{x - 1} \right]_{x \rightarrow 1}$.
5. $\left[\frac{6x^2 + 2x}{-2x^2 + 7} \right]_{x \rightarrow \infty}$.
6. $\left[\frac{x}{a} \right]_{x \rightarrow \infty}$.
7. $\left[\frac{2x^4 + 5x^3 + 7}{2x^3 + 2} \right]_{x \rightarrow \infty}$.
8. $\left[\frac{x^6 - a^6}{x - a} \right]_{x \rightarrow a}$.
9. $\left[\frac{3x^2 - 10x + 3}{3x - 1} \right]_{x \rightarrow \frac{1}{3}}$.
10. $\left[\frac{6x^2 + 2x + 1}{2x^2 - 7x + 3} \right]_{x \rightarrow \infty}$.
11. $\left[\frac{x^2 - 2x - 7}{x^3 + 5x} \right]_{x \rightarrow \infty}$.
12. $\left[\frac{(2x - 2)(3x - 3)(5x - 4)}{(x - 2)(x - 3)(x - 4)} \right]_{x \rightarrow \infty}$.
13. $\left[\frac{(x + h)^2 - x^2}{h} \right]_{h \rightarrow 0}$.
14. $\left[\frac{f(x + h) - f(x)}{h} \right]_{h \rightarrow 0}$, if $f(x) = x^4$.
15. $\left[\frac{f(x + h) - f(x)}{h} \right]_{h \rightarrow 0}$, if $f(x) = \frac{1}{x}$.
16. $\left[\frac{f(x + h) - f(x)}{h} \right]_{h \rightarrow 0}$, if $f(x) = \frac{1}{x^2}$.
17. $\left[\frac{2}{x^2 - 1} - \frac{1}{x - 1} \right]_{x \rightarrow 1}$.

CHAPTER XXV

IMAGINARY AND COMPLEX NUMBERS

ALGEBRAIC TREATMENT OF COMPLEX NUMBERS

416. Since the square root of a negative number cannot be a positive or a negative number, it becomes necessary either to exclude from our consideration such roots as $\sqrt{-a}$, $\sqrt{-1}$, or to enlarge the boundaries of the number system by the introduction of a new kind of number, called imaginary number (§ 226).

417. An **imaginary number** is an indicated square root of a negative real number; as $\sqrt{-7}$, $\sqrt{-5}$.

We assume that imaginary numbers obey the fundamental laws of algebraic operation, *i.e.* the commutative, associative, and distributive laws (§ 51).

418. Every imaginary number can be represented as the product of a real number and the quantity $\sqrt{-1}$.

For
$$\sqrt{-a} = \sqrt{a(-1)} = \sqrt{a} \sqrt{-1}.$$

419. The quantity $\sqrt{-1}$ is called the **imaginary unit**, and is usually represented by the letter i .

420. The form $a\sqrt{-1}$ or ai is said to be the **typical form** of imaginary numbers.

NOTE. In the entire chapter the letters a and b represent real numbers.

421. A **complex number** is the sum of a real number and an imaginary number.

$a + bi$, $4 + \sqrt{-3}$, are complex numbers.

422. Two complex numbers are said to be **conjugate** if they differ only in the sign of their imaginary terms.

$a + bi$ and $a - bi$ are conjugate complex numbers.

NOTE. The term *imaginary* is sometimes used in a wider sense, viz. as including imaginary and complex numbers. Numbers of the form $a\sqrt{-1}$ are then called *pure imaginaries*. Using the term in a wider sense, imaginaries may be defined as *even roots* of negative numbers.

423. Fundamental principle. From the definition of square root (§ 21), we have

$$(\sqrt{-1})^2 = -1, \text{ or } i^2 = -1.$$

NOTE. Beginners sometimes expect that $(\sqrt{-1})^2$ should equal $\sqrt{(-1)^2}$ or $\sqrt{1} = \pm 1$. The answer $+1$, however, is evidently wrong as $(+1)^2 \neq -1$. The error has been produced by a careless application of the law $(\sqrt[n]{a})^m = \sqrt[n]{a^m}$, which is true only as far as the absolute values of the two members are concerned. If this law is applied without restriction, it gives wrong results even for real numbers (§ 272).

E.g. $(\sqrt{6})^2$ equals 6, and only 6 (§ 21).

But the law applied would give

$$(\sqrt{6})^2 = \sqrt{36} = \pm 6,$$

the answer -6 being evidently wrong.

424. The **higher powers** of $\sqrt{-1}$ are found by repeated multiplication.

$$i = \sqrt{-1}.$$

$$i^2 = -1.$$

$$i^3 = -\sqrt{-1}.$$

$$i^4 = 1.$$

Since $i^4 = 1$, i^5 must equal i , $i^6 = i^2$, etc. If r denotes an integral number,

$$i^n = i^{n-4r}.$$

$$\text{E.g. } i^{27} = i^{27-24} = i^3 = -\sqrt{-1}; \quad i^{1002} = i^2 = -1.$$

425. To multiply imaginary numbers, reduce them to their typical form.

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{a} \sqrt{-1} \times \sqrt{b} \sqrt{-1} = -\sqrt{ab}.$$

NOTE. If we had applied the principle $\sqrt{m} \times \sqrt{n} = \sqrt{mn}$ directly, we should have obtained a wrong answer, viz.,

$$\sqrt{-a} \times \sqrt{-b} = \sqrt{(-a)(-b)} = \pm \sqrt{ab}.$$

The difficulty is here caused by the restriction of the sign.

If $\sqrt{-a}$ were taken in its true meaning $\pm \sqrt{-a}$, then $\pm \sqrt{-a} \times \pm \sqrt{-b}$ would equal $\pm \sqrt{ab}$.

But for practical reasons it is convenient to consider $\sqrt{-a}$ as $+\sqrt{-a}$, and then $\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$.

It should be borne in mind, however, that this difficulty is due to the arbitrary restriction of signs, and not to the exceptional nature of the imaginaries.

426. General principle. *Imaginary numbers (e.g. $\sqrt{-a^2}$) should be reduced to their typical form (i.e. $a\sqrt{-1}$) before they are added, subtracted, multiplied, divided, etc.*

427. The results of examples involving imaginaries should be represented in the typical form, $a + bi$, i.e. all real numbers should be combined and all imaginaries likewise.

428. If $a + bi = c + di$, then $a = c$, and $b = d$.

For, transposing, we have $a - c = (d - b)i$.

That is, a real number is equal to an imaginary number, which is impossible unless both members are zero.

Hence $a - c = 0$, and $d - b = 0$.

I.e. $a = c$, and $b = d$.

Ex. 1. Simplify $\sqrt{-49} + \sqrt{-64} - \sqrt{-7}$.

$$\begin{aligned} \sqrt{-49} + \sqrt{-64} - \sqrt{-7} &= 7\sqrt{-1} + 8\sqrt{-1} - \sqrt{7}\sqrt{-1} \\ &= (15 - \sqrt{7})\sqrt{-1}. \end{aligned}$$

Ex. 2. Multiply $\sqrt{-7}$ by $\sqrt{-28}$.

$$\sqrt{-7} \times \sqrt{-28} = \sqrt{7} \cdot \sqrt{-1} \cdot \sqrt{28} \cdot \sqrt{-1} = -14.$$

Ex. 3. Add $4 + \sqrt{-16}$ and $5 - \sqrt{-25}$.

$$\begin{aligned}(4 + \sqrt{-16}) + (5 - \sqrt{-25}) &= 4 + 4i + 5 - 5i \\ &= 9 - i.\end{aligned}$$

Ex. 4. Multiply $4 + \sqrt{-7}$ by $1 + \sqrt{-7}$.

$$\begin{array}{r}4 + i\sqrt{7} \\ 1 + i\sqrt{7} \\ \hline 4 + i\sqrt{7} \\ -7 + 4i\sqrt{7} \\ \hline -3 + 5i\sqrt{7}, \text{ or } -3 + 5\sqrt{-7}.\end{array}$$

Ex. 5. Divide $2 + 5\sqrt{-1}$ by $3 - 2\sqrt{-1}$.

$$\frac{2 + 5i}{3 - 2i} = \frac{2 + 5i}{3 - 2i} \times \frac{3 + 2i}{3 + 2i} = \frac{-4 + 19i}{13} = -\frac{4}{13} + \frac{19}{13}i.$$

Ex. 6. Extract the square root of $4 + 2\sqrt{-45}$.

Applying the method of § 275,

$$\text{Let} \quad \sqrt{4 + 2\sqrt{-45}} = \sqrt{x} + \sqrt{y}. \quad (1)$$

$$\text{Squaring,} \quad 4 + 2\sqrt{-45} = x + 2\sqrt{xy} + y.$$

$$\text{Hence} \quad x + y = 4. \quad (2)$$

$$2\sqrt{xy} = 2\sqrt{-45}. \quad (3)$$

$$\text{Squaring (2),} \quad x^2 + 2xy + y^2 = 16. \quad (4)$$

$$\text{Squaring (3),} \quad 4xy = -180. \quad (5)$$

$$(4) - (5), \quad x^2 - 2xy + y^2 = 196.$$

$$\text{Extracting square root,} \quad x - y = 14.$$

$$\text{But} \quad x + y = 4.$$

$$\text{Therefore} \quad x = 9, \text{ and } y = -5,$$

$$\text{and} \quad \sqrt{4 + 2\sqrt{-45}} = 3 + \sqrt{-5}.$$

The root can be found more easily by applying the inspection method (§ 276).

EXERCISE 144

Simplify:

1. $\sqrt{-25}, \sqrt{-121}, \sqrt{-a^2}, \sqrt{-b^{2n}}, \sqrt{-7}.$
2. $\sqrt{-49x^2} + \sqrt{-64x^2} - x\sqrt{-121}.$
3. $a\sqrt{-\frac{4}{9}} - \frac{1}{3}\sqrt{-25a^2} + \sqrt{-49a^2}.$
4. $\sqrt{-4x^2y^2} + x\sqrt{-9y^2} - y\sqrt{-36x^2}.$
5. $\sqrt{-3} + 2\sqrt{-3} - \sqrt{-27}.$
6. $a\sqrt{-x} - b\sqrt{-x} + \sqrt{-c^2x}.$
7. $(4 + 2\sqrt{-2}) + (3 + \sqrt{-4}) + (6 - \sqrt{-8}).$
8. $(3a + \sqrt{-16b^2}) + (8a - \sqrt{-25b^2}) + (\sqrt{-9b^2} + 2a).$
9. $\sqrt{-25x} - \sqrt{-49y} + \sqrt{-4x} - 8\sqrt{-y}.$
10. $(\sqrt{-9ab} - \sqrt{-4ab}) - (\sqrt{-ab} - \sqrt{-4ab}).$
11. $\sqrt{-8} + \sqrt{-16a} + \sqrt{-18} - \sqrt{-25a}.$
12. $(\sqrt{-1})^7 + (\sqrt{-1})^{41}.$
13. $i + i^2 + i^3 + i^4.$
14. $i^2 + 2i^7 - 6i^9 + i^{241}.$
15. $(-\sqrt{-1})^2 + (\sqrt{-1})^3 - (-\sqrt{-1})^8.$
16. $\sqrt{-3} \cdot \sqrt{-12}.$
18. $3\sqrt{-a^3} \cdot 4\sqrt{-a}.$
17. $\sqrt{-2a} \cdot \sqrt{-50a}.$
19. $\sqrt{-\frac{5a}{6b}} \cdot \sqrt{-\frac{10a}{3b}}.$
20. $(3\sqrt{-8} + \sqrt{-18} + \sqrt{-50} - 2\sqrt{-72}) \cdot \sqrt{-2}.$
21. $\sqrt{a-b} \cdot \sqrt{b-a}.$
22. $i\sqrt{-n} \cdot \sqrt{4n} + n\sqrt{-\frac{1}{2}} \cdot \sqrt{-18}.$
23. $(3 + \sqrt{-25})(3 - \sqrt{-25}).$
24. $(3 + \sqrt{-25})(4 - \sqrt{-36}).$

25. $(7 - \sqrt{-a})(6 + \sqrt{-a})$.

26. $(11 - 12i)(11 - 10i)$.

27. $(6 - \sqrt{-7a})(3 - \sqrt{-7a})$.

28. $(\sqrt{-a} + \sqrt{-b})^2$.

29. $(1 + \sqrt{-2})^2$.

30. $(\sqrt{a-2x} + \sqrt{-a+2x})^2$.

31. $\sqrt{1+i} \cdot \sqrt{1-i}$.

32. $\sqrt{-63} \div \sqrt{-9}$.

33. $\sqrt{-6} \div \sqrt{3}$.

34. $\frac{4}{1 + \sqrt{-3}}$.

35. $\frac{11}{3 - \sqrt{-2}}$.

36. $64 \div (1 + 3\sqrt{-7})$.

37. $3 \div (\sqrt{2} + \sqrt{-1})$.

47. $(\sqrt{-m} + n)^3 - (\sqrt{-m} - n)^3$.

48. $\frac{1}{1+i} + \frac{1}{1-i}$.

49. $\frac{\sqrt{-a} + \sqrt{-b}}{\sqrt{-a} - \sqrt{-b}}$.

38. $\frac{3 + 2\sqrt{-1}}{2 - 3\sqrt{-1}}$.

39. $\frac{3 - 2i}{3 + 2i}$.

40. $\frac{2 + 5i}{3i}$.

41. $\frac{1 + \sqrt{-3}}{2 - \sqrt{-3}}$.

42. $\frac{1+i}{1-i}$.

43. $\frac{\sqrt{3} + i\sqrt{2}}{\sqrt{3} - i\sqrt{2}}$.

44. $\left(\frac{1+i}{1-i}\right)^2$.

45. $(1+i)^4$.

46. $(1+i)^3 + (1-i)^3$.

50. $\frac{1 + \sqrt{-1}}{1 - \sqrt{-1}} + \frac{1 - \sqrt{-1}}{1 + \sqrt{-1}}$.

51. $\frac{3 + \sqrt{-2}}{2 - \sqrt{-1}} + \frac{3 - \sqrt{-2}}{2 + \sqrt{-1}}$.

Extract the square root of:

52. $4 - 2\sqrt{-45}$.

54. $-5 + 24i$.

56. i .

53. $3 + 4i$.

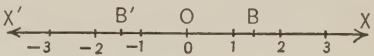
55. $2i$.

GRAPHIC REPRESENTATION OF COMPLEX NUMBERS

429. The term *imaginary* was introduced into algebra because problems whose answers involve imaginaries have usually no solutions, and it was inferred therefrom that such numbers could not exist. This view of imaginary numbers, however, is erroneous. Problems have often no solutions when the answers are real numbers; *e.g.* a problem requiring the number of persons in a room can have no solution when the answers are negative or fractional. But it would evidently be absurd to infer therefrom that negative or fractional numbers are unreal or impossible. Similarly, a problem asking for the ratio of a number of men to a number of women can have no solution when the answer is an irrational number. But this does not prove that irrational numbers are unreal.

430. To illustrate the reality of imaginary numbers, the geometrical method which was used for the representation of positive and negative numbers (§ 11) may be employed.

Let XX' be a fixed straight line, and O a fixed point in the line.

From O lay off a series of equal lengths to the right and to the left. Then any line terminating in O , as OB ,  represents a number. For

convenience we shall sometimes refer to the extremity of the line as representing the number. Thus, "point B " represents the same number as line OB .

1. Rational numbers. Rational or commensurable numbers are represented by certain points in XX' ; viz. *positive integers* by the points of division on the right of O , *negative integers* by the points of division on the left of O , and *fractions* by certain points between the points of division.

2. Irrational numbers. Not every point in XX' , however, represents an integer or a fraction, as we can construct certain

lengths which cannot be expressed by integers or fractions. Such points represent *incommensurable numbers*. *E.g.* if we lay off on OX a line OB equal to the hypotenuse of a right triangle whose other sides are equal to unity, OB represents $\sqrt{2}$, a number which cannot be equal to an integer or a fraction.



For assuming that $\frac{m}{n} = \sqrt{2}$, where m and n have no common factor, we would have $\frac{m^2}{n^2} = 2$, which is obviously impossible, as m^2 and n^2 have no common factor.

NOTE. While it is impossible to find integers or fractions which are exactly equal to an irrational number, we can find fractions which differ from the given surd by less than any number which we can assign.

Thus, $\sqrt{2}$ differs from 1.4, 1.41, 1.414 ... respectively by less than .1, .01, .001, etc.

Hence we may consider $\sqrt{2}$ the limit of the fraction 1.41421 ...

Every irrational number may be regarded as the limit of a variable rational number.

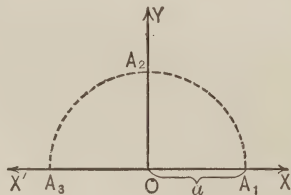
Thus, every real number is represented by a point in the line XX' , and, *vice versa*, every point in XX' represents a real number.

3. **Imaginary numbers** can be represented by points without XX' , as may be shown by the following consideration:

If A_1 be taken a units to the right of O in OX , then the line OA_1 represents the number a , and an equal line OA_3 , drawn in the opposite direction, represents $-a$.

Now obviously the line OA_1 can be brought into the position OA_3 by a rotation through an angle of 180° about O as a center,

while algebraically $+a$ is transformed into $-a$ by multiplying by -1 . Hence it follows:



431. The multiplication of a real number by -1 , is represented graphically by a rotation through an angle of 180° .

It is customary to rotate lines counter-clockwise, *i.e.* in a direction opposite to the motion of the hands of a clock.

432. To determine the algebraic meaning of a rotation through an angle less than 180° , let a rotation through an angle of 90° represent the multiplication by an unknown number x .

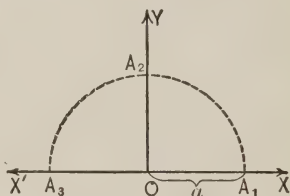
Then $OA_2 = a \cdot x$,

and $OA_3 = OA_2 \cdot x = a \cdot x \cdot x$.

Or $-a = a \cdot x^2$.

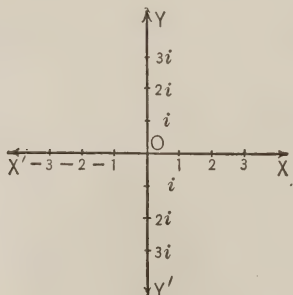
Therefore $x = \sqrt{-1}$,

and $OA_2 = a\sqrt{-1}$.



433. We may therefore consider a multiplication by $\sqrt{-1}$ to be represented by a revolution through a right angle counter-clockwise.

434. The numbers $i, 2i, 3i$, etc., may be represented respectively by the distances $1, 2, 3$, etc., laid off on OY , which is perpendicular to XX' .



Similarly, $-i, -2i, -3i$, are represented on the line OY' .

The four quantities, 3 , $3i$, -3 , $-3i$, have the same absolute value, viz. 3 , and each is represented by a line consisting of three units, but extending in different directions, viz. :

$+3$ indicating 3 units to the right,

$+3i$ indicating three units up,

-3 indicating 3 units to the left,

$-3i$ indicating 3 units down.

435. The line XX' is called the **axis of real numbers**; YY' is called the **axis of imaginaries**. The point O is called the *origin*.

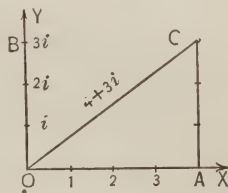
436. Graphical addition. Two real numbers, OA and OB , are added graphically by drawing from A a line AC equal to



and extending in the same direction as OB . The line OC is the required sum.

E.g. To add $+5$ and -3 , lay off $OA = 5$ to the right, and from A , lay off $AC = 3$ to the left. OC or 2 is the required sum.

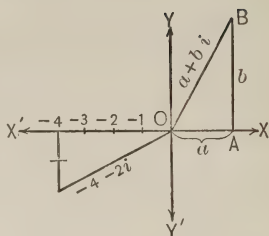
437. *Imaginaries and imaginaries, or real numbers and imaginaries, are added graphically in the same manner as real numbers.* *E.g.* to add 4 and $+3i$, lay off $OA = +4$. From A draw $AC = 3$ upward, i.e. equal and parallel to line $3i$, or OB . OC represents the required sum.



NOTE. It should be noted that $4 + 3i$ is represented by the length and direction of OC . The length of OC is 5 , but it would be erroneous to assume that $4 + 3i$ equals 5 . The number 5 is the absolute value of $4 + 3i$, but it is not equal to $4 + 3i$.

438. Complex numbers. If a and b are real numbers, the complex number $a + bi$ may be represented by OB , the sum of a and bi . *I.e.* Draw $OA = a$, and AB equal and parallel to bi . OB represents $a + bi$.

E.g. OE represents $-4 - 2i$.



439. The **absolute value** or **modulus** of any number (*i.e.* real, pure imaginary, and complex) is the length of the line which represents the number. It is always taken as positive.

The absolute value of $a + bi = OB$, or $+\sqrt{a^2 + b^2}$.

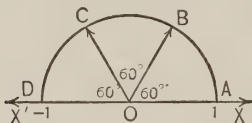
The absolute value of $-4 - 2i = \sqrt{4^2 + 2^2} = 2\sqrt{5}$.

440. The **amplitude** of OB is the angle XOB , *i.e.* the angle between OX and OB , measured from OX counter-clockwise.

Ex. 1. Determine the algebraic meaning of the rotation of a line through an angle of 60° .

Let $OA = OB = OC = OD = 1$,

and $\angle AOB = \angle BOC = \angle COD = 60^\circ$.



If x is the number which, applied as a factor, produces the required rotation,

then $OB = x$, $OC = x^2$, $OD = x^3$.

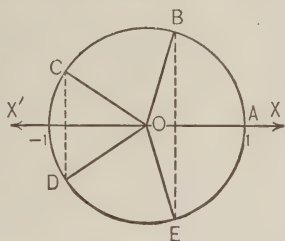
I.e. $x^3 = -1$,

or $x = \sqrt[3]{-1}$.

The rotation through an angle of 60° represents therefore a multiplication by $\sqrt[3]{-1}$, and line OB represents $\sqrt[3]{-1}$. A simple geometrical deduction shows that OB or $\sqrt[3]{-1} = \frac{1}{2} + \frac{1}{2}\sqrt{3} \cdot i$.

Ex. 2. Construct graphically the different values of $\sqrt[5]{1}$.

From O , draw five lines, OA , OB , OC , OD , and OE , each equal to 1, and forming angles of 72° with the two adjacent lines. If OA lies on the axis of real numbers, OA , OB , OC , OD , and OE represent the 5 values of $\sqrt[5]{1}$. (The proof is similar to Ex. 1.) By actual measurement we find 5 values 1 , $.31 + .95i$, $-.95 + .31i$, $-.95 - .31i$, $.31 - .95i$.



EXERCISE 145

Represent the following numbers graphically:

1. $2 + 4i$. 3. $-2 + 2i$. 5. $-2 - 3i$.

2. $4 - 3i$. 4. $i - 1$. 6. $-2i + 5$.

7. Construct $5 + 4i$ and multiply graphically by $\sqrt{-1}$.

8. Prove that the following numbers have equal absolute values: $4 + 3i$, $-3 - 4i$, -5 , and $-4 + 3i$.

9. Which has the greater absolute value, -8 or $5 - 6i$?

10. Add graphically 4 , $3i$, and -2 .

11. Add graphically -5 , $-2i$, $+2$, $+5i$.

12. Solve the equation $x^2 + 2x + 2 = 0$, and represent the roots graphically.

13. Solve the equation $x^3 - 1 = 0$, and represent the roots graphically.

14. Solve the equation $x^3 + 1 = 0$, and represent the roots graphically.

15. Divide $2i$ by $1 + i$, and represent the quotient graphically.

16. What rotation is equivalent to a multiplication by $-i$?

17. Find graphically \sqrt{i} .

18. Find one value of $\sqrt[4]{-1}$.

19. Construct 7 lines representing the values of $\sqrt[7]{1}$.

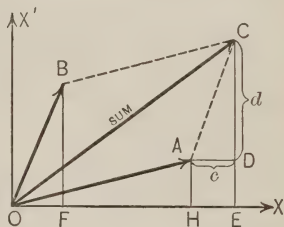
20. Represent graphically $\frac{1}{2}\sqrt{2} + \frac{i}{2}\sqrt{2}$.

21. Represent graphically $\sqrt{3} + i$.

441. Complex numbers are added graphically in the same manner as real and imaginary numbers.

E.g. to add the numbers represented by OA and OB , draw AC equal and parallel to OB ; then OC represents the required sum. (OC may also be constructed by drawing the diagonal of the parallelogram determined by OA and OB .)

If $OA = a + bi$, and $OB = c + di$, then their sum is $a + c + (b + d)i$; and in order to demonstrate that OC represents the required sum, we have to show that the coördinates of C , *i.e.*, OE and EC , are respectively equal to $a + c$ and $b + d$. If AD is drawn parallel to OX , it follows easily that $\triangle OBF = \triangle ACD$. Therefore $AD = c$, $CD = d$, and since $OH = a$ and $DE = AH = b$,



$$OE = a + c \text{ and } EC = b + d.$$

Hence

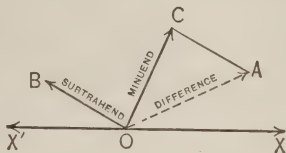
$$OC = a + c + (b + d)i.$$

NOTE. The student should bear in mind that the geometric addition of lines which have length and direction is fundamentally different from the usual addition of lines which have length only. The length of OC is obviously less than the sum of the lengths of OA and OB , but the complex number represented by OC is equal to the sum of the complex numbers represented by OA and OB . A line which has a given length and direction is called a *vector*.

442. Graphical subtraction of complex numbers.

If the number represented by OC (diagram of § 441) is the sum of the numbers represented by OA and OB , then OA is the

difference of the numbers represented by OC and OB . Hence to subtract OB from OC (see annexed diagram) draw CA equal and parallel to OB , but extending in opposite direction; then OA represents the required difference.



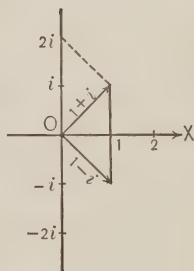
Ex. 1. Solve the equation

$$x^2 - 2x + 2 = 0,$$

and construct graphically the difference of the roots.

By formula, $x = \frac{2 \pm \sqrt{-4}}{2} = 1 \pm \sqrt{-1}.$

Draw $OA = 1 + i$, $OB = 1 - i$, and make AC equal and parallel to OB , but extending in opposite direction. $OC = 2i$ is the required difference.



Ex. 2. Solve the equation $x^6 - 64 = 0$, and add the 6 roots graphically.

Factoring, $(x - 2)(x^2 + 2x + 4)(x + 2)(x^2 - 2x + 4) = 0.$

Hence

$$x - 2 = 0,$$

$$x^2 + 2x + 4 = 0,$$

$$x + 2 = 0,$$

$$x^2 - 2x + 4 = 0.$$

These equations produce the following roots: $2, -1 \pm \sqrt{-3}, -2, 1 \pm \sqrt{-3}$, or arranged $2, 1 + i\sqrt{3}, -1 + i\sqrt{3}, -2, -1 - i\sqrt{3}, 1 - i\sqrt{3}$. Assuming $\sqrt{3} = 1.73$, we obtain respectively the 6 lines $OA_1, OA_2, OA_3, OA_4, OA_5$, and OA_6 .

8. Show graphically that the sum of two conjugate complex numbers is a real number.

9. If OA represents a given complex number $a + bi$, construct the number $2(a + bi)$.

10. Graphically subtract 1 from i , and add $-1 + i$ to the difference.

11. Draw three lines, OA , OB , and OC , representing any complex numbers, and from the sum of the first two subtract the sum of the last two.

12. Draw any five lines representing complex numbers, OA_1 , OA_2 , OA_3 , OA_4 , and OA_5 , and add them graphically.

13. If OA and OB represent two complex numbers, graphically multiply OA by i , and subtract the product from OB .

CHAPTER XXVI

PERMUTATIONS AND COMBINATIONS

443. Fundamental principle. *If one thing can be done in m different ways, and after it has been done, a second thing can be done in n different ways, then the two things can be done together in $m \times n$ different ways.*

For after the first thing is done in any one way, it can be associated with each one of the n different ways of doing the second thing, and thus each way of doing the first thing produces n ways of doing the two things together. But there are m ways of doing the first thing, hence the total number of ways of doing the two things together is $m \times n$.

Ex. 1. An alphabet contains 5 vowels and 21 consonants. How many different words of two letters can be formed from it so that the first letter is a consonant and the second a vowel?

The first letter may be selected in 21 different ways. Every one of these letters can be put together with 5 vowels, or each consonant produces 5 different words. Hence 21 consonants produce 21×5 or 105 words.

Ex. 2. Suppose there are 8 railroad lines operating between Chicago and New York, and 9 steamship lines between New York and Liverpool. In how many different ways can a man travel from Chicago to Liverpool by way of New York?

He can make the trip from Chicago to New York in 8 ways, and each of these may be combined with the 9 different ways of going to Liverpool. Hence he can make the whole journey in 8×9 or 72 different ways.

444. By successive application of § 443, it can easily be shown that the principle is true if there are three or more things each of which can be done in a given number of ways.

Ex. 3. In how many different ways can 3 persons be seated in a coach which has 4 seats?

The first person can be seated in 4 different ways. After the first person is seated, the second can be seated in 3 different ways, and finally the last person can be seated in 2 different ways. Hence the total number of ways is $4 \times 3 \times 2 = 24$.

EXERCISE 147

1. A building has 6 entrances. In how many different ways can a person enter the building and leave by a different door?

2. Between two cities 6 ferryboats are plying. In how many different ways can a man travel from one city to the other and return by a different boat?

3. A man has 5 pairs of trousers, 7 vests, and 6 coats. In how many different costumes can he appear?

4. In how many different ways may an English, a French, and a German book be selected from 6 English, 5 French, and 3 German books?

5. In how many different ways can 2 persons be seated in a coach that has 6 seats?

6. In how many different ways can 3 children be seated in 3 seats?

7. How many different words of two letters can be formed with the letters a, b, c, d , and e if the first letter is to be a vowel?

PERMUTATIONS

445. The **permutations** of a certain number of things are the different orders in which some or all of the things can be arranged.

Thus, the permutations of the letters a, b, c , taken two at a time, are ab, ac, ba, bc, ca, cb , and their permutations, taken three at a time, are $abc, acb, bac, bca, cab, cba$.

Ex. 1. Write all permutations that can be formed from the numbers 1, 2, 3, 4, all taken at a time.

1234	2134	3124	4123
1243	2143	3142	4132
1324	2314	3214	4213
1342	2341	3241	4231
1423	2413	3412	4312
1432	2431	3421	4321

Explanation. Write first all permutations whose first place is 1, then all whose first place is 2, etc. The four columns represent the four groups thus obtained. Each group, again, is divided according to the number in the second place; thus the first column contains first the permutations commencing with 12, then those commencing with 13, and last with 14. By continuing this mode of arranging the permutations, it is easy to obtain them all.

Ex. 2. Write all permutations of letters a, b, c , and d , three taken at a time.

abc	bac	cab	dab
abd	bad	cad	dac
acb	bca	cba	dba
acd	bcd	cbd	dbc
adb	bda	cda	dca
adc	bdc	cdb	dcb

446. The number of permutations of n different things taken r at a time is denoted by the symbol nP_r . Thus the number of permutations of the preceding example is 4P_3 .

This number could be obtained without writing all permutations. There are three places to be filled in by letters. The first place can be filled by a, b, c , or d , *i.e.* in 4 different ways (represented in 4 different columns). After the first place is filled the second place can be filled by one of the remaining letters, *i.e.* in 3 different ways (producing the 3 parts of each

column). Similarly, the third place can be filled in 2 different ways. Hence, according to the fundamental principle,

$${}^4P_3 = 4 \times 3 \times 2 = 24.$$

447. To find the number of permutations of n different elements taken r at a time. There are r places to be filled. The first place can be filled in any of the n ways, and after this has been filled, the second place can be filled in $(n-1)$ ways. Hence, according to the fundamental principle, the two places together can be filled in $n(n-1)$ different ways; or

$${}^nP_2 = n(n-1).$$

After the first two places are filled, the third one can evidently be filled in $(n-2)$ different ways; or

$${}^nP_3 = n(n-1)(n-2).$$

By continuing this process, it can be seen that nP_r is equal to a product of r factors, the first factor being n , and each following factor being less by one than the preceding one. Since the last factor must be $(n-r+1)$, we have

$${}^nP_r = n(n-1)(n-2) \cdots (n-r+1).$$

448. The number of permutations of n elements, taken all at a time, is

$$n(n-1)(n-2) \cdots \text{to } n \text{ factors.}$$

Or

$${}^nP_n = n(n-1)(n-2) \cdots 2 \cdot 1 = \underline{n}.$$

NOTE. Unless stated otherwise, the things are supposed to be different and not to be repeated in one permutation.

EX. 1. How many different permutations can be made by taking 4 of the letters of the word *fraction*?

$${}^8P_4 = 8 \cdot 7 \cdot 6 \cdot 5 = 1680.$$

Ex. 2. How many numbers between 1000 and 10,000 can be formed with the figures 1, 2, 3, 4, 5, 6, 7, no figure being repeated?

Since numbers between 1000 and 10,000 have four places, the required number is

$${}^7P_4 = 7 \cdot 6 \cdot 5 \cdot 4 = 840.$$

EXERCISE 148

1. Write all permutations of the numbers 1, 2, 3, taken all at a time.

2. Write all permutations of the letters a, b, c , and d , taken all at a time.

3. Write all permutations of the letters a, b, c, d , taken two at a time.

4. Write all permutations of the numbers 1, 2, 3, 4, 5, taken two at a time.

5. Find the value of 5P_4 , 6P_1 , ${}^{10}P_3$.

6. Find the value of 8P_2 , 5P_5 , 7P_4 .

7. In how many different ways can 5 pupils be seated in 5 seats?

8. In how many different ways can 5 pupils be seated in 6 seats?

9. How many different words can be formed with the letters of the word *equation*?

10. How many different numbers of three different figures can be formed from the digits 2, 3, 4?

11. How many different numbers of four different figures can be formed from the digits 1, 2, 3, 4, 5, 6?

12. How many different words of three letters can be formed from the letters a, b, c, e ?

13. In how many different ways can 6 persons be placed in 6 seats?

14. Six persons enter a car in which there are 10 seats. In how many different ways may they take their places?

15. How many different arrangements can be made by taking 3 letters of the word *theory*?

16. How many three-lettered words can be made from 10 letters, no letter being repeated in the same word?

17. In how many different ways can 5 persons form a ring?

18. If ${}^nP_3 = 4{}^nP_2$, find n .

449. To find the number of permutations of things which are not all different, taking them all at a time, let us suppose the number of permutations which can be formed by taking all the letters a, a, a, b, c , was x . If we should replace the three equal a 's by three different letters, as a_1, a_2, a_3 , the number of permutations would obviously be greater than x , for in each of the x permutations we could arrange the a, a, a in $\underline{3}$ different ways without changing the position of the b and c .

Thus, $abaac$ would produce the 6 permutations,

$$a_1ba_2a_3c, a_2ba_1a_3c, a_3ba_1a_2c, a_1ba_3a_2c, a_2ba_3a_1c, a_3ba_2a_1c.$$

Similarly, every one of the x permutations would produce $\underline{3}$ arrangements, i.e. the total number of arrangements would be $x \cdot \underline{3}$.

But, on the other hand, this number represents the number of permutations of 5 elements, all being different, or $\underline{5}$.

Hence

$$x \cdot \underline{3} = \underline{5}.$$

Therefore

$$x = \frac{\underline{5}}{\underline{3}}.$$

450. In general, the number of permutations of n things, of which p are alike, q others are alike, and r others are alike, taking them all at a time, is

$$\frac{|n|}{|p| |q| |r|}.$$

Ex. 1. In how many different ways can the letters of the word *degree* be arranged?

The word contains six letters, three of which are alike; hence the number of permutations is

$$P = \frac{|6|}{|3|} = 120.$$

Ex. 2. In how many different ways can the letters of the word *combination* be arranged?

$$P = \frac{|11|}{|2| |2| |2|} = 4989600.$$

451. *Examples, which are not special cases of the general formulæ for permutations, should be solved by means of the fundamental principle (§ 443).*

Ex. 1. In how many different ways can the letters a, b, c, d, e be arranged so that the first letter and the last shall be always consonants?

The first place can be filled in 3 different ways, and after this is done, the last place can be filled in 2 different ways. The remaining letters can in every case be arranged in $|3|$ different ways.

Hence the required number is $3 \cdot 2 \cdot |3| = 37$.

Ex. 2. A boat's crew consists of 8 men, of whom 3 can row only on the port side, and 2 only on the starboard side. Find the number of ways in which the crew can be arranged.

The first of the men who can row only on the port side can be placed in 4 ways, the second one in 3 ways, the third one in 2 ways. Similarly, the two men who can row only on the starboard side can be placed

respectively in 4 and 3 different ways. The remaining men can be placed in $\underline{3}$ ways. Hence the total number of arrangements is

$$4 \cdot 3 \cdot 2 \cdot 4 \cdot 3 \cdot \underline{3} = 1728.$$

Ex. 3. In how many different ways can the letters of the word *volume* be arranged so that the vowels occupy the even places?

The vowels can be placed respectively in 3, 2, and 1 ways. Similarly, the consonants can be placed in 3, 2, and 1 ways.

Hence the required number is $3 \cdot 2 \cdot 1 \cdot 3 \cdot 2 \cdot 1 = 36$.

Ex. 4. How many three-lettered words can be made from the letters *a, b, c, d*, and *e*, if repetitions are allowed?

The first place can be filled by 5 different letters; the second place in 5 different ways; and the third place in 5 different ways. The total number of arrangements is, therefore,

$$5 \cdot 5 \cdot 5 = 125.$$

EXERCISE 149

1. In how many different ways can 5 red balls and 3 black balls be arranged in a row?

2. How many different permutations can be formed from all the letters of the word *algebra*?

3. In how many different ways can the letters of the word *exponent* be arranged?

4. In how many ways can the letters of the word *Mississippi* be arranged?

5. Find the number of permutations that can be made from all the letters of the word *paralleliped*.

6. How many different permutations can be made from all the letters of the word *trigonometry*?

7. How many different numbers of six figures can be formed from the digits, 1, 1, 1, 2, 2, 3?

8. In how many different ways can 4 quarters, 2 dollars, and 3 dimes be arranged in a line?

9. In how many different ways can 3 white and 4 blue flags be displayed at a time, one above the other?

10. How many permutations can be made from all the letters of the word *electricity*?

11. How many different permutations can be made from the letters of the word *radium* so that the vowels occupy the even places?

12. A stage accommodates 5 passengers on each side. In how many different ways can 6 persons be seated if 2 of them always take their seats on one side, and a third on the other side?

13. On a railroad there are 20 stations. How many different tickets are required to connect every station with every other one?

14. How many different arrangements beginning with the letter *f* and ending with a consonant can be formed out of the letters of the word *factor*?

15. How many different words of three letters can be formed out of the letters *a, b, c, e, u*, when the letters may be repeated?

16. How many numbers of five figures can be formed with the digits 1, 2, 3, 4, if the digits may be repeated?

17. A railway signal has 3 different arms, and each arm may be placed in 4 different positions. Find the total number of signals that can be made.

COMBINATIONS

452. The combinations of n things taken r at a time are the different groups of r things that can be selected from the n things without regard to their order.

Thus, the combinations of the letters *a, b, c*, taken two at a time are *ab, ac, bc*. The arrangements *ab* and *ba* represent different permutations, but the same combination.

Ex. Write all combinations that can be formed from the letters a, b, c, d, e , taken three at a time.

Write first all combinations containing a . Of these, take first all which contain ab , etc. In this manner we obtain $abc, abd, abe, acd, ace, ade, bcd, bce, bde, cde$.

453. To find the number of combinations of n different things taken r at a time.

Let x denote the required number of combinations. Since each of the combinations contains r different things, we could arrange these things in $\lfloor r$ different ways, or each combination would produce $\lfloor r$ permutations. Hence the total number of permutations that could be formed would be $x \cdot \lfloor r$. But, on the other hand, this number represents the total number of permutations that can be formed from the n things taken r at a time, or nP_r .

$$\text{Hence} \quad x \cdot \lfloor r = {}^nP_r = n(n-1) \cdots (n-r+1).$$

$$\text{Therefore} \quad x = \frac{n(n-1) \cdots (n-r+1)}{\lfloor r}.$$

454. This number was represented by the symbol nC_r (§ 374). Hence we may represent the number of combinations of n things taken r at a time by nC_r , and

$${}^nC_r = \frac{n(n-1)(n-2) \cdots (n-r+1)}{\lfloor r}. \quad (1)$$

455. If we multiply numerator and denominator of this formula by $\lfloor n-r$ (§ 377), we obtain

$${}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}. \quad (2)$$

Hence (§ 377)

$${}^nC_r = {}^nC_{n-r}. \quad (3)$$

NOTE. In formula (2), if $n = r$,

$${}^nC_n = \frac{[n]}{[n] [0]}.$$

But

$${}^nC_n = 1, \text{ hence } [0] = 1.$$

Similarly,

$${}^nC_0 = {}^nC_n = 1.$$

Ex. 1. In how many different ways can a baseball nine be selected from 11 candidates ?

This is evidently the number of combinations of 11 elements taken 9 at a time.

But
$${}^{11}C_9 = {}^{11}C_2 = \frac{11 \cdot 10}{1 \cdot 2} = 55.$$

Ex. 2. Out of the first 10 letters of the alphabet how many different selections of 4 letters may be made so as to include the letter a ?

The a can be taken only one way ; the remaining letters can be selected in ${}^{10}C_3$ different ways. Hence the total number of selections is

$${}^{10}C_3 = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120.$$

Ex. 3. Out of 5 Republicans and 6 Democrats, how many different committees can be formed, each consisting of 3 Republicans and 4 Democrats ?

The number of ways in which the Republicans can be selected is 5C_3 , while the Democrats can be chosen in 6C_4 different ways. Since each of the first group can be combined with each of the second group, the total number of selections is

$${}^5C_3 \times {}^6C_4 = {}^5C_2 \times {}^6C_2 = \frac{5 \cdot 4}{1 \cdot 2} \times \frac{6 \cdot 5}{1 \cdot 2} = 150.$$

Ex. 4. How many words, each containing 2 consonants and 3 vowels, can be formed from 6 consonants and 7 vowels ?

The consonants can be chosen in 6C_2 , the vowels in 7C_3 , different ways. Hence ${}^6C_2 \times {}^7C_3$ different sets of letters may be selected. The letters of each selection can be arranged in 5P_5 different ways. Hence the required number is

$${}^6C_2 \times {}^7C_3 \times {}^5P_5 = 15 \times 35 \times 120 = 63,000.$$

Ex. 5. How many different sums of money can be made up by combining any number of the following 5 coins : a cent, a dime, a half-dollar, a dollar, and a five-dollar piece ?

The cent may be disposed of in 2 different ways, *i.e.* it may be either included or not included in the sum. Similarly, each coin can be disposed of in 2 different ways, hence all can be disposed of in $2 \times 2 \times 2 \times 2 \times 2 = 2^5$ different ways. But among these is one case in which none of the coins is taken, a case which evidently has to be excluded. Hence the required number is

$$2^5 - 1, \text{ or } 31.$$

EXERCISE 150

1. Write all combinations of the letters a, b, c, d, e , taken two at a time.
2. Write all combinations of the letters of the word *surd*, taken three at a time.
3. Write all combinations of the letters of the word *ratio*, taken three at a time.
4. Find the value of ${}^{10}C_9, {}^{10}C_0, {}^5C_4$.
5. If ${}^nC_2 = 15$, find n .
6. If ${}^nC_{n-2} = 10$, find n .
7. How many different combinations of three letters can be formed from the 26 letters of the alphabet ?
8. How many different selections of 20 letters may be formed from 23 letters ?
9. In how many different ways can a committee of 3 be selected from a class of 21 students ?
10. How many different straight lines are determined by 12 points, no three of which are in a straight line ?
11. How many diagonals can be drawn in a decagon ?
12. How many different planes are determined by 9 points, no four of which lie in a plane ?

13. How many different triangles are determined by 11 points, no three of which lie in the same straight line?
14. Find the greatest number of points in which 6 straight lines may intersect.
15. In how many different ways can a guard of 5 men be formed from a detachment of 10 soldiers?
16. In how many different ways can 4 letters be selected from the letters of the word *equation*, if each selection has to contain the letter *e*?
17. From 4 teachers and 20 students, how many different committees can be selected, each consisting of 2 teachers and 3 students?
18. How many words can be formed by selecting 2 consonants and 3 vowels from 5 vowels and 20 consonants?
19. From a detachment of 20 soldiers, in how many different ways can a guard of 6 be selected, when 2 particular men are always excluded?
20. If ${}^nC_4 = {}^nC_{11}$, find n .

EXERCISE 151

MISCELLANEOUS EXAMPLES

1. If the number of permutations of n things, taken 3 at a time, is equal to 12 times the number of combinations of n things taken two at a time, find n .
2. From 20 soldiers and 10 sailors, how many different parties of 3 soldiers and 2 sailors can be formed?
3. If ${}^nC_4 = {}^nC_6$, find nC_7 .
4. How many different numbers greater than 52,000 can be formed from the figures 1, 2, 3, 4, 5?

5. How many different significant numbers can be formed with the digits 0, 1, 2, 3, if each figure may be repeated?

6. A boat's crew consists of 8 men, of whom 2 can row only on the port side and 1 only on the starboard side. Find the number of ways in which the crew can be arranged.

7. In how many different ways can the word *algebra* be read in the following arrangement of letters, if we commence at the first letter of the first line and advance from letter to letter either by a downward step or a step to the right?

```

a l g e b r a
l g e b r a
g e b r a
e b r a
b r a
r a
a

```

8. How many signals can be made with 8 different flags, if 5 of them are displayed at a time, one above another?

9. How many signals can be made with 4 different flags, if any number of them may be displayed at a time, one above another?

10. In how many different ways can 3 dollars, 2 half-dollars, and 4 dimes be placed in a circle?

11. How many arrangements commencing and ending with a consonant can be made from the letters *a, b, c, d, e*?

12. How many numbers between 4000 and 6000 can be made with the digits 4, 5, 7, 8?

13. How many different sums can be made up by combining any number of the following coins: a cent, a nickel, a dime, a quarter, a half-dollar, and a dollar?

14. Find n , if ${}^nP_4 = 6 {}^nP_3$.

15. From 6 teachers, 10 boys, and 10 girls, how many committees, each consisting of 2 teachers, 2 boys, and 2 girls, can be selected?

16. How many different throws can be made with 3 dice?

17. What is the greatest number of points in which 8 straight lines can intersect?

18. In how many different ways can 7 presents be given to 2 children so that one receives 3, the other 4, presents?

19. How many different numbers of 5 figures can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, so that the first, third, and fifth digits are odd, if no digit is repeated?

CHAPTER XXVII

DETERMINANTS

DETERMINANTS OF THE SECOND ORDER

456. Consider the simultaneous equations

$$a_1x + b_1y = c_1, \quad (1)$$

$$a_2x + b_2y = c_2. \quad (2)$$

Solving, we obtain

$$x = \frac{c_1b_2 - c_2b_1}{a_1b_2 - a_2b_1}, \quad y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \quad (3)$$

The common denominator may be written in the form

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

This symbol is called a **determinant**, and it signifies the difference of the cross products of the four quantities involved.

Thus,
$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1, \quad (4)$$

and
$$\begin{vmatrix} 4 & 5 \\ 2 & 3 \end{vmatrix} = 4 \cdot 3 - 2 \cdot 5 = 2.$$

457. The quantities a_1, b_1, a_2, b_2 in the determinant $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ are the **elements** of the determinant.

458. A **row** is a horizontal line of elements, a **column** a vertical line of elements.

The rows in (4) are a_1b_1 and a_2b_2 .

The columns are $\begin{smallmatrix} a_1 \\ a_2 \end{smallmatrix}$ and $\begin{smallmatrix} b_1 \\ b_2 \end{smallmatrix}$.

459. The expression $a_1b_2 - a_2b_1$ is said to be the **expansion** of the determinant.

The **terms** of a determinant are the terms of its expansion; as, a_1b_2 .

460. The **order** of a determinant is the number of terms in a row or a column.

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is a determinant of the second order.

The **principal diagonal** is composed of the elements from the upper left-hand corner to the lower right-hand corner; as, a_1b_2 .

The line a_2b_1 is called the *secondary diagonal*.

461. The roots of equations (1) and (2) may be written

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

I.e. the common denominator is the determinant composed of the coefficients of x and y .

The numerator of the value of x may be formed from the denominator by substituting c_1 and c_2 in place of the corresponding coefficients of x (*i.e.* a_1, a_2).

The numerator of y may be formed from the denominator by substituting c_1 and c_2 in place of the coefficients of y .

Ex. Solve the system $\begin{cases} 4x - 2y = 6, \\ 2x + 3y = 7. \end{cases}$

$$x = \frac{\begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}}{\begin{vmatrix} 4 & -2 \\ 2 & 3 \end{vmatrix}} = \frac{32}{16} = 2.$$

$$y = \frac{\begin{vmatrix} 4 & 6 \\ 2 & 7 \end{vmatrix}}{\begin{vmatrix} 4 & -2 \\ 2 & 3 \end{vmatrix}} = \frac{16}{16} = 1.$$

EXERCISE 152

Expand:

$$1. \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}, \quad 3. \begin{vmatrix} -1 & -2 \\ 3 & -5 \end{vmatrix}, \quad 5. \begin{vmatrix} 1 & a \\ 1 & b \end{vmatrix}, \quad 7. \begin{vmatrix} 1 & a \\ a & -a^2 \end{vmatrix},$$

$$2. \begin{vmatrix} 5 & -1 \\ 7 & 6 \end{vmatrix}, \quad 4. \begin{vmatrix} 2 & 0 \\ 7 & -3 \end{vmatrix}, \quad 6. \begin{vmatrix} a & 3a \\ b & 2b \end{vmatrix}, \quad 8. \begin{vmatrix} a & x \\ 0 & 0 \end{vmatrix}.$$

$$9. \begin{vmatrix} 3 & x \\ 2 & y \end{vmatrix}, \quad 10. \begin{vmatrix} a+b & a+c \\ a-c & a-b \end{vmatrix}.$$

Solve:

$$11. \begin{cases} 4x - y = 13, \\ 5x - y = 17. \end{cases}$$

$$15. \begin{cases} 2x + 5y = 17a, \\ 3x - 7y = 5a. \end{cases}$$

$$12. \begin{cases} 3x + y = 9, \\ 2x - 2y = -2. \end{cases}$$

$$16. \begin{cases} mx - ny = 0, \\ cx - dy = 1. \end{cases}$$

$$13. \begin{cases} x + 2y = 5, \\ x + 3y = 7. \end{cases}$$

$$17. \begin{cases} mx + ny = p, \\ ax + by = c. \end{cases}$$

$$14. \begin{cases} x + y = a + 1, \\ x - y = a - 2. \end{cases}$$

$$18. \begin{cases} ax - by = c, \\ ax + by = d. \end{cases}$$

Prove the following identities:

$$19. \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}.$$

$$21. \begin{vmatrix} ma_1 & b_1 \\ ma_2 & b_2 \end{vmatrix} = m \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.$$

$$20. \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

$$22. \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0, \text{ if } a:b=c:d.$$

$$23. a:b=c:d, \text{ if } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0.$$

Express as determinants:

$$24. 4 \cdot 5 - 3 \cdot 2. \quad 25. 6 \cdot 7 - 3(-2). \quad 26. 22 - 17. \quad 27. a^2 - b^2.$$

DETERMINANTS OF THE THIRD ORDER

462. In any term whose literal factors follow in natural order, the occurrence of a larger subscript before a smaller one is called an **inversion**.

Thus, $a_1b_3c_2$ contains one inversion, viz. b_3c_2 .

$a_3b_1c_2$ contains two inversions, viz. a_3b_1, a_3c_2 .

$a_3b_2c_1$ contains three inversions, viz. a_3b_2, a_3c_1, b_2c_1 .

463. If we solve the three simultaneous equations

$$\begin{cases} a_1x + b_1y + c_1z = d_1, & (1) \end{cases}$$

$$\begin{cases} a_2x + b_2y + c_2z = d_2, & (2) \end{cases}$$

$$\begin{cases} a_3x + b_3y + c_3z = d_3, & (3) \end{cases}$$

we obtain for x

$$x = \frac{d_1b_2c_3 - d_1b_3c_2 + d_2b_3c_1 - d_2b_1c_3 + d_3b_1c_2 - d_3b_2c_1}{a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1}. \quad (4)$$

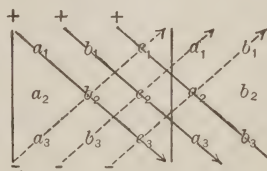
The denominator is usually written as a **determinant of the third order**, or

$$a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \quad (5)$$

464. A determinant of the third order is equal to the sum of all the products that can be formed by taking one element, and only one, from each column and from each row, considering the sign of each product positive if it contains an even number of inversions, and negative if it contains an odd number of inversions.

Thus, $a_1b_3c_2$ contains one inversion, hence it is negative ;
 $a_2b_3c_1$ contains two inversions, hence it is positive ;
 $a_3b_2c_1$ contains three inversions, hence it is negative.

465. The annexed figure offers a convenient method for expanding a determinant of the third order, but it should be borne in mind that this method does not apply to determinants of a higher order than the third.



466. By grouping the terms of the expansion according to a_1 , a_2 , and a_3 , and denoting the determinant (5) by D , we obtain

$$D = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1),$$

or
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}. \quad (6)$$

The coefficients of a_1 , $-a_2$, and a_3 are called the *minors* of these elements.

467. The **minor** of any element is the determinant obtained by erasing the row and the column which contains the element.

Thus the minor of b_1 is obtained by erasing the first row and the second column, i.e.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ or } \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}.$$

468. If an element occurs in the m th row and in the n th column, its minor multiplied by $(-1)^{m+n}$ is called its **co-factor**.

Thus, the co-factor of a_2 is $-\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$.

Thus, the co-factor of c_3 is $+\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

469. The co-factors of a_1, a_2, a_3, b_1 , etc., are usually denoted by A_1, A_2, A_3, B_1 , etc.

Thus, $B_2 = \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$; $C_2 = -\begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$.

470. Using this notation, (6) may be written

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 A_1 + a_2 A_2 + a_3 A_3.$$

471. By grouping the terms of the expansion according to the elements of the second row (*i.e.* a_2, b_2, c_2), we have

$$D = -a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - c_2 \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix},$$

or

$$D = a_2 A_2 + b_2 B_2 + c_2 C_2.$$

Similarly for any row or column. Hence we have in general:

472. *A determinant of the third order is equal to the sum of the products obtained by multiplying the elements of any row or column by their respective co-factors.*

473. The evaluating of determinants by means of co-factors is usually more convenient than by direct expansion.

Ex. 1. Find the numerical value of

$$\begin{vmatrix} 4 & 3 & 2 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{vmatrix}.$$

Expanding in terms of the elements of the first row, we have

$$\begin{aligned} & 4 \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} - 3 \begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} \\ &= 4 \cdot 1 - 3 \cdot (-10) + 2 \cdot (-7) = 20. \end{aligned}$$

Ex. 2. Find the numerical value of

$$\begin{vmatrix} 2 & -1 & 3 \\ 4 & 1 & 0 \\ 2 & -2 & 1 \end{vmatrix}.$$

Since the third column contains a zero, expand in terms of the elements of the third column.

$$\begin{aligned} & 3 \begin{vmatrix} 4 & 1 \\ 2 & -2 \end{vmatrix} - 0 \begin{vmatrix} 2 & -1 \\ 2 & -2 \end{vmatrix} + 1 \begin{vmatrix} 2 & -1 \\ 4 & 1 \end{vmatrix} \\ &= 3 \cdot (-10) - 0 + 6 = -24. \end{aligned}$$

Ex. 3. Expand $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}.$

Expanding in terms of the elements of the first row, we have

$$\begin{aligned} & (b+c) \begin{vmatrix} c+a & b \\ c & a+b \end{vmatrix} - a \begin{vmatrix} b & b \\ c & a+b \end{vmatrix} + a \begin{vmatrix} b & c+a \\ c & c \end{vmatrix} \\ &= (b+c)(a^2+ab+ac) - a(ab+b^2-bc) + a(bc-c^2-ac) \\ &= a^2b+ab^2+abc+a^2c+abc+ac^2-a^2b-ab^2+abc+abc-ac^2-a^2c \\ &= 4abc. \end{aligned}$$

EXERCISE 153

1. Determine the number of inversions in $a_1b_3c_2d_4$, $a_4b_3c_1d_2$, $a_4b_1c_3d_2$, and $a_3b_4c_3d_2e_1$.

2. In the expansion of $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & b_3 \end{vmatrix}$, what is the sign of $a_3b_1c_2$, $a_2b_1c_3$, $a_3b_2c_1$?

3. Expand $\begin{vmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ m_3 & n_3 & p_3 \end{vmatrix}$.

4. In the determinant of Ex. 3, what is the minor (a) of n_2 ? (b) of p_2 ? (c) of m_3 ?

5. In the same determinant what is the co-factor of (a) n_1 ? (b) of m_2 ? (c) of n_3 ?

6. In the expansion of $\begin{vmatrix} x & y & z \\ 1 & 2 & 3 \\ 4 & 5 & -1 \end{vmatrix}$, find the coefficient of x .

7. In the same determinant, what is the coefficient of y ?

8. Expand $\begin{vmatrix} x & 2 & 1 \\ y & 3 & 2 \\ z & -1 & 2 \end{vmatrix}$.

Find the value of the following determinants:

$$9. \begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 1 \\ 2 & 4 & 6 \end{vmatrix} \quad 10. \begin{vmatrix} 2 & 0 & 1 \\ 3 & 2 & 2 \\ 4 & 1 & -2 \end{vmatrix} \quad 11. \begin{vmatrix} 4 & 0 & 0 \\ 7 & 2 & 3 \\ 8 & 2 & 4 \end{vmatrix}$$

$$12. \begin{vmatrix} 1 & a & a \\ a & 3 & 2 \\ a & 1 & a \end{vmatrix} \quad 14. \begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$$

$$13. \begin{vmatrix} 4 & 9 & -3 \\ 2 & 1 & -1 \\ 1 & 7 & 2 \end{vmatrix} \quad 15. \begin{vmatrix} 0 & 4 & 5 \\ 4 & 0 & 2 \\ 5 & 2 & 0 \end{vmatrix}$$

Solve the equations:

$$16. \begin{vmatrix} 4 & x & x \\ 2 & -1 & 3 \\ 1 & 2 & -1 \end{vmatrix} = 0.$$

$$17. \begin{vmatrix} 2 & 3 & 1 \\ 3 & -1 & 2 \\ 1+x & 1-x & x \end{vmatrix} = 0.$$

Express as determinants of the third order:

$$18. m_1 \begin{vmatrix} b_2 & 1 \\ b_3 & 1 \end{vmatrix} - m_2 \begin{vmatrix} b_1 & 1 \\ b_3 & 1 \end{vmatrix} + m_3 \begin{vmatrix} b_1 & 1 \\ b_2 & 1 \end{vmatrix}.$$

$$19. a(n-p) - b(m-p) + c(m-n).$$

DETERMINANTS OF THE FOURTH ORDER

474. The principles derived for determinants of the third order are also applicable to determinants of the fourth and higher orders.

475. Thus the symbol

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

is a **determinant of the fourth order**, and it is equal to *the sum of all the products that can be formed by taking one, and only one, element from each column and from each row, considering the sign of each product positive if it contains an even number of inversions, negative if it contains an odd number of inversions.*

476. Since the terms of the expansion contain the same literal factors, but differ in the arrangement of the subscripts, we may obtain all terms by forming all permutations of the subscripts 1, 2, 3, and 4.

Hence the total number of terms is 4P_4 , i.e. $\underline{4}$, or 24.

477. If all the terms containing a_1 are combined, the coefficient of a_1 cannot contain any elements of the first row or the first column, but it must consist of the sum of all products that can be formed by taking one element, and only one, from the last three columns and last three rows. The signs of these products are positive or negative according as the number of inversions is even or odd.

Hence the coefficient of a_1 is the minor of a_1 , i.e.

$$\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ d_4 & c_4 & b_4 \end{vmatrix}.$$

If the terms containing a_2 were collected, we should obtain in a similar manner

$$-a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}.$$

Collecting the terms containing a_3 and a_4 , we have

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

Evidently the coefficient of every element is the co-factor of the element. Hence, using the same notation as in the preceding chapter, we have

$$D = a_1 A_1 + a_2 A_2 + a_3 A_3 + a_4 A_4.$$

478. In a similar manner the determinant may be expanded in terms of the elements of any row or any column.

E.g. the expansion in terms of the elements of the third row is

$$a_3 A_3 + b_3 B_3 + c_3 C_3 + d_3 D_3.$$

479. If all elements in a row or a column are zero, the determinant equals zero.

For expanding in terms of the elements of that row or column, each term becomes zero.

480. If all the elements *but one* in a row or a column are equal to zero, the determinant is equal to the product of the element and its co-factor, and hence reduces to a determinant of the next lower order; *e.g.*

$$\begin{vmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & d \\ a_4 & b_4 & c_4 & 0 \end{vmatrix} = -d \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{vmatrix}.$$

Ex. 1. Find the numerical value of

$$\begin{vmatrix} 4 & 2 & 1 & 1 \\ 1 & 0 & 2 & 3 \\ 2 & -1 & 2 & 0 \\ 3 & 2 & 0 & 1 \end{vmatrix}.$$

Expanding in terms of the elements of the second row,

$$\begin{aligned} & -1 \begin{vmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 2 & 0 & 1 \end{vmatrix} + 0 - 2 \begin{vmatrix} 4 & 2 & 1 \\ 2 & -1 & 0 \\ 3 & 2 & 1 \end{vmatrix} + 3 \begin{vmatrix} 4 & 2 & 1 \\ 2 & -1 & 2 \\ 3 & 2 & 0 \end{vmatrix} \\ &= -1(+1) + 0 - 2(-1) + 3(+3) \\ &= -1 + 2 + 9 = 10. \end{aligned}$$

Ex. 2. Find the coefficient of x in the expansion of

$$\begin{vmatrix} a & x & y & n \\ 2 & 5 & 1 & 3 \\ 2 & -11 & 0 & 2 \\ 4 & 27 & 2 & 1 \end{vmatrix}.$$

The coefficient is

$$\begin{aligned}
 - \begin{vmatrix} 2 & 1 & 3 \\ 2 & 0 & 2 \\ 4 & 2 & 1 \end{vmatrix} &= + \begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} - 0 + 2 \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} \\
 &= -6 - 4 = -10.
 \end{aligned}$$

EXERCISE 154

1. In the expansion of the determinant find the coefficient of x .
- $$\begin{vmatrix} 4 & 2 & x & 1 \\ 2 & -1 & y & 0 \\ 4 & -2 & z & 2 \\ 1 & a & 2 & 6 \end{vmatrix},$$
2. In the same determinant find the coefficient of y .
3. In the same determinant find the coefficient of z .
4. In the same determinant find the coefficient of a .

Find the value of the following determinants:

$$5. \begin{vmatrix} 4 & 2 & -1 & 17 \\ 2 & 9 & 0 & 6 \\ 1 & 1 & 0 & 2 \\ 2 & 3 & 0 & 4 \end{vmatrix}.$$

$$8. \begin{vmatrix} a & 0 & 0 & 133 \\ 2 & 0 & a & 77 \\ 0 & 0 & 0 & a \\ y & a & x & 77 \end{vmatrix}.$$

$$6. \begin{vmatrix} 4 & 5 & 19 & 7 \\ 0 & 0 & 0 & 0 \\ 2 & 1 & 7 & 2 \\ 4 & 5 & 6 & 7 \end{vmatrix}.$$

$$9. \begin{vmatrix} 2 & 1 & 2 & 4 \\ 0 & 0 & 3 & 0 \\ 4 & 2 & -3 & 2 \\ -2 & 1 & 0 & 2 \end{vmatrix}.$$

$$7. \begin{vmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{vmatrix}.$$

$$10. \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{vmatrix}.$$

11. Reduce to a determinant of the third order

$$\begin{vmatrix} a & b & 0 & c \\ d & e & 0 & f \\ g & h & i & k \\ l & m & 0 & p \end{vmatrix}.$$

12. Reduce to a determinant of the second order

$$\begin{vmatrix} a & 0 & 0 & 0 \\ x & y & b & c \\ m & 0 & n & x \\ q & 0 & r & s \end{vmatrix}.$$

GENERAL PROPERTIES OF DETERMINANTS

481. *The value of a determinant is not altered when the rows are changed to columns and the columns to rows.*

Obviously $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$; i.e. the proposition is true for determinants of the second order.

$$\text{Let } D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } D' = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

Expanding D in terms of the elements of the first column, and D' in terms of the elements of the first row, we have

$$D = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$$

$$D' = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

Hence $D = D'$.

Similarly for determinants of higher orders.

482. Any theorem which is true for the columns of a determinant is true for its rows, and vice versa.

483. The interchange of any two adjacent columns (or rows) of a determinant changes the sign of the determinant.

$$\text{E.g. let } D = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} \text{ and } D' = \begin{vmatrix} a_1 & c_1 & b_1 & d_1 \\ a_2 & c_2 & b_2 & d_2 \\ a_3 & c_3 & b_3 & d_3 \\ a_4 & c_4 & b_4 & d_4 \end{vmatrix}.$$

The minor of b_1 in D is evidently equal to the minor of b_1 in D' , but their respective co-factors have opposite signs. Hence, denoting the co-factor of b_1 in D by B_1 , the co-factor of b_1 in D' is $-B_1$, etc.

$$\text{Hence} \quad D = b_1 B_1 + b_2 B_2 + b_3 B_3 + b_4 B_4,$$

$$\text{and} \quad D' = -b_1 B_1 - b_2 B_2 - b_3 B_3 - b_4 B_4.$$

$$\text{Therefore} \quad D = -D'.$$

484. The interchange of any two columns (or rows) of a determinant changes the sign of the determinant.

For the interchange of any two columns can be effected by an odd number of successive changes of adjacent columns. *E.g.* to interchange the 1st and 4th columns make three successive interchanges of adjacent columns, viz. 1st and 2d, 2d and 3d, 3d and 4th. The former 4th column is now the 3d, hence two successive changes will carry it to the first place. Or the total number of exchanges is 5, and $(-1)^5 = -1$. Similarly, if m interchanges are necessary to bring a particular column in place of any other one, $m-1$ will bring the second one in place of the first, or the total number of interchanges is $2m-1$; i.e. an odd number.

$$\text{E.g.} \quad \begin{vmatrix} a_2 & a & 1 \\ b_2 & b & 1 \\ c_2 & c & 1 \end{vmatrix} = - \begin{vmatrix} 1 & a & a_2 \\ 1 & b & b_2 \\ 1 & c & c_2 \end{vmatrix}.$$

485. *If two columns (or rows) of a determinant are identical, the determinant vanishes; i.e. it is equal to zero.*

Let
$$D = \begin{vmatrix} a_1 & b_1 & a_1 & c_1 \\ a_2 & b_2 & a_2 & c_2 \\ a_3 & b_3 & a_3 & c_3 \\ a_4 & b_4 & a_4 & c_4 \end{vmatrix}.$$

By the preceding paragraph the interchange of the two identical columns changes the sign of the determinant; i.e. the resultant determinant equals $-D$.

But, on the other hand, an interchange of two identical columns does not change the determinant.

Hence
$$D = -D, \text{ or } 2D = 0.$$

Therefore
$$D = 0.$$

E.g.
$$\begin{vmatrix} 6 & x & 6 \\ 7 & y & 7 \\ a & z & a \end{vmatrix} = 0.$$

486. *If the elements of any column (or row) be multiplied by the co-factors of the corresponding elements of any other column (or row), the sum of the products vanishes.*

For such a sum represents a determinant with two identical rows or columns.

E.g.
$$b_1A_1 + b_2A_2 + b_3A_3 = \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Similarly, $a_2A_1 + b_2B_1 + c_2C_1 = 0.$

487. *If all elements in any column are multiplied by any factor, the determinant is multiplied by that factor.*

$$\begin{aligned}
 \text{For } \begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix} &= ma_1A_1 + ma_2A_2 + ma_3A_3 \\
 &= m(a_1A_1 + a_2A_2 + a_3A_3) \\
 &= m \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.
 \end{aligned}$$

This principle is also true for a division, for a division by m is equivalent to a multiplication by $\frac{1}{m}$.

$$\text{Thus, } \begin{vmatrix} ma_1 & a_1 & a_2 \\ mb_1 & b_1 & b_2 \\ mc_1 & c_1 & c_2 \end{vmatrix} = m \begin{vmatrix} a_1 & a_1 & a_2 \\ b_1 & b_1 & b_2 \\ c_1 & c_1 & c_2 \end{vmatrix} = 0.$$

$$\text{Or } \begin{vmatrix} 33 & 30 \\ 66 & 75 \end{vmatrix} = 33 \cdot 15 \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 33 \cdot 15 \cdot 1 = 495.$$

488. *If each element of any column (or row) of a determinant is the sum of two quantities, the determinant can be expressed as the sum of two determinants of the same order.*

$$\begin{aligned}
 \text{For } \begin{vmatrix} a_1 + d_1 & b_1 & c_1 \\ a_2 + d_2 & b_2 & c_2 \\ a_3 + d_3 & b_3 & c_3 \end{vmatrix} &= (a_1 + d_1)A_1 + (a_2 + d_2)A_2 + (a_3 + d_3)A_3 \\
 &= (a_1A_1 + a_2A_2 + a_3A_3) + (d_1A_1 + d_2A_2 + d_3A_3) \\
 &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}.
 \end{aligned}$$

$$\text{E.g. } \begin{vmatrix} 4 + 1 & 4 \\ 5 + 2 & 5 \end{vmatrix} = \begin{vmatrix} 4 & 4 \\ 5 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix}.$$

489. *If each element of a column is multiplied by the same number, and the products be added to (or subtracted from) the corresponding elements of another column, the determinant is not altered.*

According to the preceding paragraph we have

$$\begin{vmatrix} a_1 \pm mb_1 & b_1 & c_1 \\ a_2 \pm mb_2 & b_2 & c_2 \\ a_3 \pm mb_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \pm \begin{vmatrix} mb_1 & b_1 & c_1 \\ mb_2 & b_2 & c_2 \\ mb_3 & b_3 & c_3 \end{vmatrix}.$$

But the last determinant equals

$$m \begin{vmatrix} b_1 & b_1 & c_1 \\ b_2 & b_2 & c_2 \\ b_3 & b_3 & c_3 \end{vmatrix} \text{ and hence vanishes (§ 485).}$$

Ex. Evaluate $\begin{vmatrix} 4244 & 4245 \\ 4246 & 4247 \end{vmatrix}.$

The determinant = $\begin{vmatrix} 4244 & 4245 \\ 2 & 2 \end{vmatrix} = \begin{vmatrix} 4244 & 1 \\ 2 & 0 \end{vmatrix} = -2.$

490. Evaluation of determinants. By means of the preceding proposition all elements but one in a column (or row) can be made equal to zero, and hence the determinant can be reduced to one of the next lower order (§ 480). In many cases, however, the determinant, before reduction to a lower order, should be simplified as follows:

(1) Remove factors common to all elements in a row or a column.

(2) Diminish the absolute values of the elements by subtracting the corresponding elements of other columns (or rows) or multiples of these elements.

Ex. 1. Evaluate
$$\begin{vmatrix} 410 & 86 & 51 \\ 420 & 88 & 57 \\ 300 & 60 & 120 \end{vmatrix}.$$

Remove from the three columns respectively the factors 10, 2, 3, and from the last row the factor 10. The result is

$$600 \begin{vmatrix} 41 & 43 & 17 \\ 42 & 44 & 19 \\ 3 & 3 & 4 \end{vmatrix}.$$

Subtracting the first column from the second, we have

$$600 \begin{vmatrix} 41 & 2 & 17 \\ 42 & 2 & 19 \\ 3 & 0 & 4 \end{vmatrix}.$$

Subtracting the first row from the second,

$$\begin{aligned} 600 \begin{vmatrix} 41 & 2 & 17 \\ 1 & 0 & 2 \\ 3 & 0 & 4 \end{vmatrix} &= -600 \cdot 2 \cdot \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \\ &= -1200 (-2) \\ &= 2400. \end{aligned}$$

Ex. 2. Evaluate
$$\begin{vmatrix} 3 & 2 & -2 & 3 \\ 4 & 3 & 7 & -2 \\ 5 & 1 & 2 & 3 \\ 6 & 2 & -3 & 1 \end{vmatrix}.$$

From the first row subtract the third; to the second add twice the fourth; from the third subtract three times the fourth. The result is

$$\begin{vmatrix} -2 & 1 & -4 & 0 \\ 16 & 7 & 1 & 0 \\ -13 & -5 & 11 & 0 \\ 6 & 2 & -3 & 1 \end{vmatrix}, \text{ which reduces to } \begin{vmatrix} -2 & 1 & -4 \\ 16 & 7 & 1 \\ -13 & -5 & 11 \end{vmatrix}.$$

To the first column add twice the second; to the third column add four times the second:

$$\begin{vmatrix} 0 & 1 & 0 \\ 30 & 7 & 29 \\ -23 & -5 & -9 \end{vmatrix}. \quad \text{This reduces to}$$

$$-1 \begin{vmatrix} 30 & 29 \\ -23 & -9 \end{vmatrix} = -1 \begin{vmatrix} 7 & 20 \\ -23 & -9 \end{vmatrix} = -397.$$

Ex. 3. Evaluate $\begin{vmatrix} -a+b+c & 2a & 1 \\ a-b+c & 2b & 1 \\ a+b-c & 2c & 1 \end{vmatrix}.$

Adding the second column to the first, the determinant becomes

$$= \begin{vmatrix} a+b+c & 2a & 1 \\ a+b+c & 2b & 1 \\ a+b+c & 2c & 1 \end{vmatrix} = 2(a+b+c) \begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix} = 0.$$

Ex. 4. Evaluate $D = \begin{vmatrix} bc & \frac{1}{a} & a \\ ac & \frac{1}{b} & b \\ ab & \frac{2}{c} & c \end{vmatrix}.$

Dividing the determinant by abc , and multiplying the three rows respectively by a , b , and c , we have

$$\begin{aligned} D &= \frac{1}{abc} \begin{vmatrix} abc & 1 & a^2 \\ abc & 1 & b^2 \\ abc & 2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & a^2 \\ 1 & 1 & b^2 \\ 1 & 2 & c^2 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & a^2 \\ 1 & 0 & b^2 \\ 1 & 1 & c^2 \end{vmatrix} = -1 \begin{vmatrix} 1 & a^2 \\ 1 & b^2 \end{vmatrix} = a^2 - b^2. \end{aligned}$$

491. Factoring of a determinant. If a determinant vanishes when any element b is substituted for another element a , then $a - b$ is a factor of the determinant (§ 288).

$$\text{Ex. 1. Factor } D = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$$

If $a = b$, the resulting determinant will contain two identical rows, and hence will vanish.

Therefore $a - b$ is a factor of D .

For similar reasons $(b - c)$ and $(c - a)$ are factors.

Since the product of these three factors is of the same degree as D , $(a - b)(b - c)(c - a)$ and D can differ only in a numerical factor, *e.g.* K .

$$\text{Therefore} \quad D = K(a - b)(b - c)(c - a).$$

The term obtained from the principal diagonal in D is bc^2 . This term must be equal to the similar term of the right member, *viz.* Kbc^2 .

$$\text{Hence} \quad bc^2 = Kbc^2. \quad \text{Or } K = 1.$$

$$\text{Therefore} \quad D = (a - b)(b - c)(c - a).$$

$$\text{Ex. 2. Factor } D = \begin{vmatrix} a & -n & a \\ x & b & y \\ x & -c & y \end{vmatrix}.$$

If $x = y$, or $b = -c$, the determinant will contain two identical columns, or rows, and hence vanish.

If $a = 0$, D is reduced to a determinant of the second order, which also vanishes.

Since D is of the third degree, we have

$$D = K(a - 0)(x - y)(b + c).$$

$$\text{Equating two terms, } aby = -Kaby, \text{ i.e. } K = -1.$$

$$\text{Therefore} \quad D = -a(x - y)(b + c).$$

EXERCISE 155

Find the value of the following determinants:

$$1. \begin{vmatrix} 67800 & 6781 \\ 67820 & 6783 \end{vmatrix}.$$

$$7. \begin{vmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 1 \end{vmatrix}.$$

$$2. \begin{vmatrix} a & 7 & a \\ b & x & b \\ c & 19 & c \end{vmatrix}.$$

$$8. \begin{vmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{vmatrix}.$$

$$3. \begin{vmatrix} 4 & 9 & 2 \\ 2 & 0 & 2 \\ 6 & 3 & -1 \end{vmatrix}.$$

$$9. \begin{vmatrix} 1111 & 90 \\ 9999 & 900 \end{vmatrix}.$$

$$4. \begin{vmatrix} 10 & 120 & 15 \\ 10 & 130 & 2 \\ 10 & 140 & 9 \end{vmatrix}.$$

$$10. \begin{vmatrix} 10 & 4 & 0 \\ 20 & 12 & 6 \\ 30 & 8 & 12 \end{vmatrix}.$$

$$5. \begin{vmatrix} 4a & x & a \\ 4b & y & b \\ 4c & z & c \end{vmatrix}.$$

$$11. \begin{vmatrix} 1 & 2 & 2 \\ 3 & -1 & 2 \\ 6 & 1 & 1 \end{vmatrix}.$$

$$6. \begin{vmatrix} 5a & 0 & 20 \\ 5b & 2 & 30 \\ 5c & 3 & 10 \end{vmatrix}.$$

$$12. \begin{vmatrix} 101 & 102 & 103 \\ 104 & 105 & 106 \\ 107 & 108 & 110 \end{vmatrix}.$$

$$13. \begin{vmatrix} 1 & b & a \\ a & 1 & b \\ b & a & 1 \end{vmatrix}.$$

$$15. \begin{vmatrix} a+b & c & c \\ a & b+c & a \\ b & b & c+a \end{vmatrix}.$$

$$17. \begin{vmatrix} 1 & a & b \\ a & 4 & b \\ a & b & 2 \end{vmatrix}.$$

$$14. \begin{vmatrix} 0 & a^2 & b \\ b^2 & 0 & a^2 \\ a & b^2 & 0 \end{vmatrix}.$$

$$16. \begin{vmatrix} a & b & c \\ d & a & b \\ e & d & a \end{vmatrix}.$$

$$18. \begin{vmatrix} a & 2 & a & -7 \\ b & 5 & b & 2 \\ c & 7 & c & 3 \\ d & 9 & d & 4 \end{vmatrix}.$$

$$19. \begin{vmatrix} x & a & 3x & -11 \\ y & b & 3y & -12 \\ z & c & 3z & -13 \\ u & d & 3u & -14 \end{vmatrix}.$$

$$20. \begin{vmatrix} 4 & 5 & 6 & 0 \\ 4 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 17 & 5 & 11 & 2 \end{vmatrix}.$$

$$21. \begin{vmatrix} 1 & 2 & 3 & 1 \\ 2 & 3 & 1 & 2 \\ 3 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \end{vmatrix}.$$

$$22. \begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 2 \\ 3 & 1 & 2 & 1 \\ 1 & 2 & 2 & 2 \end{vmatrix}.$$

$$23. \begin{vmatrix} 4 & 3 & -1 & -2 \\ -1 & -2 & 2 & 1 \\ 3 & 4 & -3 & 4 \\ 1 & 2 & 3 & -1 \end{vmatrix}.$$

Resolve into factors :

$$24. \begin{vmatrix} a & a & a \\ x & x & b \\ b & c & d \end{vmatrix}.$$

$$26. \begin{vmatrix} ab & 1 & c \\ bc & 1 & a \\ ca & 1 & b \end{vmatrix}.$$

$$28. \begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}.$$

$$25. \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & a^2 \\ 1 & b & b^2 \end{vmatrix}.$$

$$27. \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+b \end{vmatrix}.$$

$$29. \begin{vmatrix} a & x & a & a \\ b & b & y & b \\ c & c & c & z \\ d & d & d & d \end{vmatrix}.$$

Prove the following identities :

$$30. \begin{vmatrix} x & b & c \\ x & b & y \\ y & b & c \end{vmatrix} = b(c-y)(x-y).$$

$$31. \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1+a & 1 & 1 \\ 1 & 1 & 1+b & 1 \\ 1 & 1 & 1 & 1+c \end{vmatrix} = abc.$$

$$32. \begin{vmatrix} x+a & x+2a & x+3a \\ x+2a & x+3a & x+4a \\ x+3a & x+4a & x+5a \end{vmatrix} = 0.$$

$$33. \begin{vmatrix} 1 & 1 & b & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & 1 & c \\ 1 & 1 & 1 & 1 \end{vmatrix} = (a-1)(b-1)(c-1).$$

$$34. \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

$$35. \begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$$

Find the value of the following determinants:

$$36. \begin{vmatrix} n^2 & (n+1)^2 & (n+2)^2 \\ (n+1)^2 & (n+2)^2 & (n+3)^2 \\ (n+2)^2 & (n+3)^2 & (n+4)^2 \end{vmatrix}. \quad 38. \begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ba & b^2+bc & c^2 \end{vmatrix}.$$

$$37. \begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix}.$$

Simplify the following expressions:

$$39. (a_1 + b_1 + c_1)(A_1 + B_1 + C_1) + (a_2 + b_2 + c_2)(A_2 + B_2 + C_2) + (a_3 + b_3 + c_3)(A_3 + B_3 + C_3), \text{ if } a, b, c \text{ are the elements of a determinant of the third order, and } A, B, C \text{ their respective co-factors.}$$

$$40. \begin{vmatrix} a+bi & a+ci \\ a-ci & a-bi \end{vmatrix}, \text{ if } i = \sqrt{-1}.$$

$$41. \begin{vmatrix} i & i & 5 \\ 1 & 4 & 2i \\ 2 & 2 & i \end{vmatrix}, \text{ if } i = \sqrt{-1}.$$

SOLUTION OF LINEAR SIMULTANEOUS EQUATIONS AND ELIMINATION

492. Consider the simultaneous equations :

$$a_1x + b_1y + c_1z = d_1, \quad (1)$$

$$a_2x + b_2y + c_2z = d_2, \quad (2)$$

$$a_3x + b_3y + c_3z = d_3. \quad (3)$$

Let
$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \quad (4)$$

Multiplying the first three equations respectively by A_1 , A_2 , and A_3 (*i.e.* the co-factors of a_1 , a_2 , a_3), we have

$$a_1A_1x + b_1A_1y + c_1A_1z = d_1A_1, \quad (5)$$

$$a_2A_2x + b_2A_2y + c_2A_2z = d_2A_2, \quad (6)$$

$$a_3A_3x + b_3A_3y + c_3A_3z = d_3A_3. \quad (7)$$

Considering that $b_1A_1 + b_2A_2 + b_3A_3 = 0$,

and $c_1A_1 + c_2A_2 + c_3A_3 = 0$,

we obtain by addition of (5), (6), and (7),

$$(a_1A_1 + a_2A_2 + a_3A_3)x = d_1A_1 + d_2A_2 + d_3A_3.$$

Hence
$$x = \frac{d_1A_1 + d_2A_2 + d_3A_3}{a_1A_1 + a_2A_2 + a_3A_3}.$$

That is,

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

In a similar manner, we obtain:

$$y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

493. It should be noted that (§ 461):

The common denominator of the three roots is the determinant composed of the coefficients of x , y , and z .

The numerator of the value of x may be obtained from the denominator by substituting d_1 , d_2 , and d_3 in place of the corresponding coefficients of x .

Similarly, the numerators of y and z are formed by substituting respectively d_1 , d_2 , and d_3 in place of the corresponding coefficients of y or z .

Ex. Solve the system

$$x - 2y + 3z = 2,$$

$$2x - 3z = 3,$$

$$x + y + z = 6.$$

$$D = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 0 & -3 \\ 1 & 1 & 1 \end{vmatrix} = 19.$$

$$\text{Hence } x = \frac{\begin{vmatrix} 2 & -2 & 3 \\ 3 & 0 & -3 \\ 6 & 1 & 1 \end{vmatrix}}{19}, \quad y = \frac{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & -3 \\ 1 & 6 & 1 \end{vmatrix}}{19}, \quad z = \frac{\begin{vmatrix} 1 & -2 & 2 \\ 2 & 0 & 3 \\ 1 & 1 & 6 \end{vmatrix}}{19}.$$

Whence $x = 3$, $y = 2$, $z = 1$.

494. The roots of a system of four or more simultaneous equations can be obtained in exactly the same manner.

495. Elimination. Eliminate x and y from the three equations:

$$a_1x + b_1y + c_1 = 0, \quad (1)$$

$$a_2x + b_2y + c_2 = 0, \quad (2)$$

$$a_3x + b_3y + c_3 = 0. \quad (3)$$

Multiplying the equations in order by C_1 , C_2 , and C_3 , the respective co-factors of c_1 , c_2 , and c_3 in the determinant,

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad (4)$$

and adding,

$$(a_1C_1 + a_2C_2 + a_3C_3)x + (b_1C_1 + b_2C_2 + b_3C_3)y + c_1C_1 + c_2C_2 + c_3C_3 = 0.$$

$$\text{Or, considering § 486,} \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \quad (5)$$

This equation expresses the condition which must be satisfied by the coefficients of a_1 , b_1 , c_1 , etc., if the given equations are consistent.

496. The left member of equation (5) is the **eliminant** of the given system of equations.

497. If the eliminant is a quantity which cannot be equal to zero, the equations are inconsistent.

Thus, $4x + 5 = 0$ and $3x + 2 = 0$ are inconsistent, since $\begin{vmatrix} 4 & 5 \\ 3 & 2 \end{vmatrix} \neq 0$.

498. Similarly, we may eliminate x , y , and z from the following system:

$$a_1x + b_1y + c_1z + d_1 = 0,$$

$$a_2x + b_2y + c_2z + d_2 = 0,$$

$$a_3x + b_3y + c_3z + d_3 = 0,$$

$$a_4x + b_4y + c_4z + d_4 = 0.$$

The result is

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

Ex. 1. Eliminate x and y from the following system :

$$2x + y + a = 0,$$

$$x - 2y + 3 = 0,$$

$$x + y + a = 0.$$

The eliminant is

$$\begin{vmatrix} 2 & 1 & a \\ 1 & -2 & 3 \\ 1 & 1 & a \end{vmatrix} = 0.$$

Or, expanding,

$$-2a - 3 = 0.$$

Ex. 2. Eliminate x and y from the equations :

$$3x + 2y - z = 3,$$

$$2x - 2y + 3z = 6,$$

$$x - 2z = -3.$$

Transposing the right members and applying § 498, we have

$$\begin{vmatrix} 3 & 2 & -z-3 \\ 2 & -2 & 3z-6 \\ 1 & 0 & -2z+3 \end{vmatrix} = 0.$$

Expanding and simplifying, $z - 2 = 0$.

Ex. 3. Eliminate x from the two equations :

$$x^2 + xy + y^2 = a^2, \tag{1}$$

$$x - 2y = -2a. \tag{2}$$

Transposing the right members and multiplying (2) by x , we obtain the three equations

$$x^2 + xy + y^2 - a^2 = 0,$$

$$x - 2y + 2a = 0,$$

$$x^2 - x(2y - 2a) = 0.$$

Eliminating,

$$\begin{vmatrix} 1 & y & y^2 - a^2 \\ 0 & 1 & -2y + 2a \\ 1 & -2y + 2a & 0 \end{vmatrix} = 0.$$

Expanding,

$$7y^2 - 10ay + 3a^2 = 0.$$

EXERCISE 156

Solve the equations:

$$1. \begin{cases} 2x + y + z = 4, \\ x + 2y + 3z = 6, \\ 3x + y + 5z = 9. \end{cases}$$

$$5. \begin{cases} x - 2y + z = 0, \\ 7x + 6y + 7z = 100, \\ 3x + y - 2z = 0. \end{cases}$$

$$2. \begin{cases} x + y - z = 17, \\ x + z - y = 13, \\ -x + y + z = 7. \end{cases}$$

$$6. \begin{cases} x + y + z = 26, \\ 7x - 11z = 0, \\ 9y - 14x = 0. \end{cases}$$

$$3. \begin{cases} x + 2y + 4z = 15, \\ x + y + z = 9, \\ x + 3y + 9z = 23. \end{cases}$$

$$7. \begin{cases} x + y + z = 5, \\ 3x - 5y + 7z = 75, \\ 9x - 11z + 10 = 0. \end{cases}$$

$$4. \begin{cases} x + y - z = 0, \\ x + 2y + z = 8, \\ 3x + 5y - z = 10. \end{cases}$$

$$8. \begin{cases} x + y = 16, \\ y + z = 28, \\ z + x = 22. \end{cases}$$

$$9. \begin{cases} x + y + z = a + b + c, \\ ax + by + cz = ab + ac + bc, \\ (b - c)x + (c - a)y + (a - b)z = 0. \end{cases}$$

$$10. \begin{cases} x + y + z + u = 60, \\ x + 2y + 3z + 4u = 100, \\ x + 3y + 6z + 10u = 150, \\ x + 4y + 10z + 20u = 210. \end{cases}$$

11. Eliminate x from the following equations:

$$\begin{cases} 523x + 498a = 0, \\ 522x + 497a = 0. \end{cases}$$

12. Eliminate x and y from the three equations:

$$\begin{cases} ax + by - 2 = 0, \\ x - y - 1 = 0, \\ x + y - a = 0. \end{cases}$$

13. Eliminate x and y from the following equations, and factor the result:

$$\begin{cases} x + ay + a^2 = 0, \\ x + by + b^2 = 0, \\ x + cy + c^2 = 0. \end{cases}$$

14. Eliminate x , y , and z from the following equations:

$$\begin{cases} ax + y + z + 1 = 0, \\ x + ay = 0, \\ y + z = 0, \\ x + y + z + a = 0. \end{cases}$$

15. Eliminate x and y from the following three equations:

$$\begin{cases} x + y + z = 9, \\ x + 2y + 3z = 14, \\ x + 3y + 7z = 21. \end{cases}$$

16. Eliminate y and z from the following equations:

$$\begin{cases} (a + b)x + (a - b)z = 2bc, \\ (b + c)y + (b - c)x = 2ac, \\ (c + a)z + (c - a)y = 2ab. \end{cases}$$

17. Eliminate x, y, z from the following equations:

$$\begin{cases} 3x + y + z = 20, \\ x + 4y + 3u = 30, \\ 6x + z + 3u = 40, \\ 8y + 3z + 5u = 50. \end{cases}$$

18. Eliminate x from the following equations:

$$\begin{cases} 2x^2 - 2y^2 = 2, \\ 3x + 2y = 2. \end{cases}$$

Are the following equations consistent?

$$19. \begin{cases} x + 6y - 5 = 0, \\ 2x - 3y - 1 = 0, \\ x + y - 2 = 0. \end{cases}$$

$$20. \begin{cases} 3x + 8y - 22 = 0, \\ 2x - 3y - 2 = 0, \\ x + y - 6 = 0. \end{cases}$$

$$21. \begin{cases} x^2 + 3x - 7 = 0, \\ 2x - 3 = 0. \end{cases}$$

$$22. \begin{cases} 2x^2 + 2x + 1 = 0, \\ x^2 + x + 2 = 0. \end{cases}$$

HINT. Multiply each equation by x , and eliminate x^3, x^2 , and x .

CHAPTER XXVIII

THEORY OF EQUATIONS

499. General form of an equation of the n th degree. Every equation of the n th degree can be reduced to the form :

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0. \quad (1)$$

Dividing every term by a_0 , and substituting p for $\frac{a}{a_0}$, we obtain the equation of the n th degree in its *simplest form* :

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0. \quad (2)$$

Unless stated otherwise, the coefficients $p_1, p_2, \cdots p_n$ are supposed to be rational.

500. If none of the coefficients $p_1, p_2, \cdots p_n$ is zero, the equation is **complete**, otherwise the equation is **incomplete**.

$x^3 - 3x^2 + 4x - 7$ is a complete cubic equation.

$x^5 - 2x^4 + 9$ is an incomplete equation.

501. In the present chapter we shall usually represent the *rational integral function* $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$ by $J(x)$. Thus equation (2) may be written $f(x) = 0$.

502. A **commensurable** root is a real root which is either integral or fractional. Roots which cannot be expressed exactly by integers or fractions are called **incommensurable** roots.

SYNTHETIC DIVISION

503. The work of the following chapters requires frequently the division of a rational integral function by a binomial of

the form $x - a$. In such cases the work of division can be greatly shortened by a method called **synthetic division**.

Divide $3x^3 - 4x^2 - 23x + 9$ by $x - 4$.

$$\begin{array}{r|l}
 3x^3 - 4x^2 - 23x + 9 & x - 4 \\
 \underline{3x^3 - 12x^2} & 3x^2 + 8x + 9 \\
 + 8x^2 - 23x & \\
 + 8x^2 - 32x & \\
 \hline
 & + 9x + 9 \\
 & + 9x - 36 \\
 & \hline
 & + 45
 \end{array}$$

This work may be abridged by omitting the literal factors (Appendix I. — Detached Coefficients).

$$\begin{array}{r|l}
 3 - 4 - 23 + 9 & 1 - 4 \\
 \underline{3 - 12} & 3 + 8 + 9 \\
 + 8 - 23 & \\
 + 8 - 32 & \\
 \hline
 & + 9 + 9 \\
 & + 9 - 36 \\
 & \hline
 & + 45
 \end{array}$$

The second term of each partial product may be omitted, for it is merely a repetition of the number above, and by changing the sign of the second term in the divisor, every subtraction may be changed to an addition.

$$\begin{array}{r|l}
 3 - 4 - 23 + 9 & 1 + 4 \\
 \underline{3 + 12} & 3 + 8 + 9 \\
 + 8 & \\
 \hline
 & + 32 \\
 & + 9 \\
 \hline
 & + 36 \\
 & \hline
 & + 45
 \end{array}$$

Since the first term of each successive remainder is equal to the corresponding term of the quotient, we may omit the quotient. Omitting also the first term of the divisor, the work can be contracted as follows :

$$\begin{array}{r|l}
 3 - 4 - 23 & + 9 \quad | \quad + 4 \\
 + 12 + 32 & + 36 \\
 \hline
 \text{Quotient} = 3 + 8 + 9; & + 45 = \text{Remainder.}
 \end{array}$$

504. To divide, therefore, any rational integral function $f(x)$ by $x - a$, proceed as follows:

Write the coefficients of x in a horizontal line, representing all missing powers of x by zero coefficients; and bring down the first coefficient.

Multiply the first coefficient by a , and add the product to the second coefficient.

Multiply the resulting sum by a , and add the product to the next coefficient, and so forth.

The last sum is the remainder, and the preceding numbers in order are the coefficients of the quotient.

Ex. 1. Divide $2x^4 - 45x^2 - 9x - 7$ by $x - 5$.

$$\begin{array}{r} 2 + 0 - 45 - 9 - 7 \quad | + 5 \\ + 10 + 50 + 25 + 80 \\ \hline 2 + 10 + 5 + 16; + 73 \end{array}$$

The quotient is $2x^3 + 10x^2 + 5x + 16$, the remainder $+ 73$.

Ex. 2. Divide $3x^4 + 3x^3 + 7$ by $x + 2$.

$$\begin{array}{r} 3 + 3 + 0 + 0 + 7 \quad | - 2 \\ - 6 + 6 - 12 + 24 \\ \hline \end{array}$$

Quotient $= 3 - 3 + 6 - 12; + 31 =$ Remainder.

EXERCISE 157

By synthetic division, find the quotient and the remainder of each of the following divisions:

- $(x^4 - 2x^3 + x^2 - 2x + 1) \div (x - 2)$.
- $(2x^4 - 3x^3 - 5x^2 - 2x + 7) \div (x - 3)$.
- $(x^4 - 3x^2 - x + 3) \div (x - 1)$.
- $(2x^4 + 3x^3 - 2x^2 + 5x - 7) \div (x + 3)$.
- $(3x^4 - 2x^2 + x - 6) \div (x - 4)$.
- $(3x^5 - 2x^4 + 2x^3 - 3x^2 + 2x + 2) \div (x + 2)$.

7. $(2x^5 - 3x^3 + 2x - 1) \div (x - 4)$.
8. $(x^4 - 6x^3 + 4x^2 - 2x - 7) \div (x - 5)$.
9. $(x^4 + 6x^3 + 4x^2 + 2x + 7) \div (x + 5)$.
10. $(x^6 - 12) \div (x - 1)$.
11. $(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1) \div (x - 1)$.
12. $(x^6 + 3x^2 - 9) \div (x + 2)$.
13. $(6x^4 - 5x^3 + 2x^2 - 9x - 7) \div (x - 3)$.
14. $(x^3 - 2ax^2 + 2a^2x + a^3) \div (x - a)$.

APPLICATION OF THE FACTOR THEOREM

505. If $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$, and a is a root of the equation $f(x) = 0$, then $f(x)$ is divisible by $x - a$. (Factor Theorem.)

The proof which was given in Chapter XVI may be briefly restated as follows:

Divide $f(x)$ by $(x - a)$ until the remainder R no longer involves x . Denoting the quotient by $Q(x)$, we have

$$(x - a)Q(x) + R = f(x). \quad (1)$$

Substituting a for x , $R = 0$.

I.e. $f(x)$ is divisible by $x - a$.

E.g. if $f(x) = x^4 - 5x^3 + x^2 + 2x + 1$, then $f(1) = 0$. Hence $x - 1$ is a factor of $f(x)$.

506. Conversely, if a rational integral function $f(x)$ is divisible by $x - a$, then a is a root of the equation $f(x) = 0$. (1)

For if $Q(x)$ is the quotient obtained by dividing $f(x)$ by $x - a$, equation (1) becomes

$$(x - a)Q(x) = 0.$$

This equation is obviously satisfied by $x = a$.

Ex. 1. Prove that 5 is a root of the equation

$$x^3 - 3x^2 - 18x + 40 = 0.$$

$$\begin{array}{r} \text{Dividing by } x - 5, \quad 1 - 3 - 18 + 40 \quad \underline{5} \\ + 5 + 10 - 40 \\ \hline 1 + 2 - 8; \quad 0 \end{array}$$

Since there is no remainder, 5 is a root of the given equation.

507. If $f(x)$ is divided by $x - a$, the remainder of the division is equal to $f(a)$. (Remainder Theorem, Chapter XVI.)

Using the notation of § 505, we have

$$(x - a)Q(x) + R = f(x).$$

$$\text{Substituting } x = a, \quad R = f(a).$$

Ex. 2. If $f(x) = x^4 - 2x^3 - 9x^2 + 2$, find $f(4)$.

Dividing $f(x)$ by $x - 4$,

$$\begin{array}{r} 1 - 2 - 9 + 0 + 2 \quad \underline{4} \\ + 4 + 8 - 4 - 16 \\ \hline 1 + 2 - 1 - 4; \quad -14 \end{array}$$

$$\text{Hence} \quad f(4) = -14.$$

508. The preceding *method of substitution* may also be demonstrated as follows:

$$\begin{array}{r} x^4 - 2x^3 - 9x^2 + 0 \cdot x + 2 \quad \underline{4} \\ + 4x^3 + 8x^2 - 4x - 16 \\ \hline x^4 + 2x^3 - x^2 - 4x; \quad -14 \end{array}$$

I.e. if $x = 4$,

$$x^4 = 4x^3, \text{ which added to the next term gives } +2x^3.$$

$$2x^3 = 8x^2, \text{ which added to the next term gives } -x^2.$$

$$-x^2 = -4x, \text{ which added to the next term gives } -4x.$$

$$-4x = -16, \text{ which added to the next term gives } -14.$$

$$\text{Therefore} \quad f(4) = -14.$$

EXERCISE 158

1. Prove that $x - 1$ is a factor of $66x^5 - 66x^4 + 73x^3 - 70x^2 + 4x - 7$.
2. Prove that $x - 1$ is a factor of $637x^{11} - 638x^{12} + 1$.
3. Prove that $x - a$ is a factor of $6x^{14} - 8x^7a^7 - 4x^2a^{12} + 6a^{14}$.
4. Prove that $a + 2b$ is a factor of $a^4 - 16b^4$.

Prove the following statements by synthetic division:

5. -2 is a root of $x^5 - 2x^4 - 3x^3 + 3x^2 - 12x + 4 = 0$.
6. 3 is a root of $x^4 + x^3 - 17x^2 + 17x - 6 = 0$.
7. -2 is a root of $x^6 + 4x^5 + 7x^4 + 10x^3 + 13x^2 + 16x + 12 = 0$.
8. -5 is a root of $x^4 + 7x^3 + 7x^2 - 11x + 20 = 0$.
9. 3 is a root of $3x^4 - 6x^3 - 2x^2 - 20x - 3 = 0$.
10. -5 is a root of $2x^4 + 13x^3 + 20x^2 + 31x + 30 = 0$.
11. -6 is a root of $4x^5 + 19x^4 - 27x^3 + 20x^2 + 13x + 6 = 0$.
12. 3 is a root of $5x^5 - 18x^4 + 11x^3 - 10x^2 + 13x - 3 = 0$.
13. -5 is a root of $6x^5 + 15x^4 - 55x^3 + 64x^2 - 136x + 220 = 0$.
14. If $f(x) = 3x^3 + 2x^2 - 3x - 5$, find $f(2)$.
15. If $f(x) = 2x^3 - 3x^2 + 4x - 7$, find $f(-2)$.
16. If $f(x) = 4x^4 - 5x^3 + 2x^2 - 7x - 9$, find $f(3)$.
17. If $f(x) = 6x^6 - 2x^3 - 7$, find $f(-1)$.
18. If $f(x) = 4x^4 - 3x^2 - 7x + 2$, find $f(6)$.
19. If $f(x) = 6x^5 - 18x^4 + 3x^3 - 10x - 2$, find $f(3)$.
20. If $f(x) = x^4 - 5ax^3 + 2a^2x^2 - 17a^3x - 41a^4$, find $f(3a)$.

NUMBER OF ROOTS

509. We shall assume the fundamental theorem, that every rational, integral equation has at least one root. The proof of this proposition is beyond the scope of this book.*

510. *Every rational, integral equation of the n th degree has n roots.*

If $f(x) = x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_{n-1}x + p_n$, and r_1 is a root of the equation $f(x) = 0$, then $f(x)$ is divisible by $x - r_1$ (§ 505).

Denoting $f(x) \div (x - r_1)$ by $f_1(x)$, we have

$$f(x) = (x - r_1)f_1(x).$$

But $f_1(x)$ is a rational, integral function of the $(n - 1)$ degree, hence it has a root. Let this root be r_2 , then we obtain in a similar manner as before $f_1(x) = (x - r_2)f_2(x)$, where $f_2(x)$ is of $(n - 2)$ degree.

Therefore $f(x) = (x - r_1)(x - r_2)f_2(x)$.

Continuing this process, $f(x)$ can be resolved in n factors, viz. $x - r_1, x - r_2 \dots x - r_n$.

Or $f(x) = (x - r_1)(x - r_2)(x - r_3) \dots (x - r_n)$.

Hence the equation $f(x)$ has n roots, for $f(x)$ vanishes when x is equal to any of the values $r_1, r_2, r_3 \dots r_n$.

NOTE. If $F(x) = a_0x^n + a_1x^{n-1} \dots a_n$, it can easily be shown that

$$F(x) = a_0(x - r_1)(x - r_2)(x - r_3) \dots (x - r_n).$$

511. Multiple roots. The n roots of an equation of the n th degree are not necessarily all different. *E.g.* the equation $(x - 4)(x - 4)(x - 3) = 0$ has the roots 4, 4, 3. Every root occurring more than once is called a *multiple* root; thus 4 is a *double* root.

In such cases, however, the roots are counted as if they were all different.

* See Burnside and Panton, *Theory of Equations*, p. 259.

512. Depression of an equation. If r_1 is a root of the equation $f(x)=0$, we can, by dividing $f(x)$ by $x-r_1$, reduce or depress the equation to one of the next lower degree, which contains all the remaining roots.

E.g. the equation $x^3-5x^2-9x+45=0$ has one root equal to 3. Dividing by $x-3$, we obtain the depressed equation $x^2-2x-15=0$, whose roots are 5 and -3 .

Hence 5, -3 , and 3 are the roots of the given equation.

513. Solution by trial. If all roots but two of an equation are integers, it is often possible to solve the equation by trial.

Ex. Solve $x^5+4x^4-3x^3-32x^2-54x-36=0$.

If there are integral roots, they must be factors of 36; *i.e.* $\pm 1, \pm 2, \pm 3$, etc. (§ 525).

Substituting $+1$, $1+4-3-32-54-36 \neq 0$.

Substituting -1 , $-1+4+3-32+54-36 \neq 0$.

Hence $+1$ and -1 are not roots.

Dividing by $x-2$,

$$\begin{array}{r} 1 \quad +4 \quad -3 \quad -32 \quad -54 \quad -36 \quad \underline{2} \\ +2 \quad +12 \quad +18 \quad -28 \quad -164 \\ \hline 1 \quad +6 \quad +9 \quad -14 \quad -82; \quad -200 \end{array}$$

Hence 2 is not a root.

Dividing by $x+2$,

$$\begin{array}{r} 1 \quad +4 \quad -3 \quad -32 \quad -54 \quad -36 \quad \underline{-2} \\ -2 \quad -4 \quad 14 \quad +36 \quad +36 \\ \hline 1 \quad +2 \quad -7 \quad -18 \quad -18; \quad 0 \end{array}$$

Therefore, -2 is a root, and the depressed equation is

$$x^4+2x^3-7x^2-18x-18=0.$$

As -2 may be a double root, we divide again by $x+2$, but obtain a remainder.

Dividing by $x-3$,

$$\begin{array}{r} 1 \quad +2 \quad -7 \quad -18 \quad -18 \quad \underline{3} \\ +3 \quad +15 \quad +24 \quad +18 \\ \hline 1 \quad +5 \quad +8 \quad +6; \quad 0 \end{array}$$

That is, $x=3$ is another root, and the next quotient is x^3+5x^2+8x+6 .

Dividing by $x + 3$,

$$\begin{array}{r} 1 \quad +5 \quad +8 \quad +6 \quad | \quad -3 \\ -3 \quad -6 \quad -6 \quad \\ \hline 1 \quad +2 \quad +2 \quad ; \quad 0 \end{array}$$

Therefore, $x = -3$ is a third root, and the last quotient is $x^2 + 2x + 2$, which cannot be factored.

Solving $x^2 + 2x + 2 = 0$ by formula, we obtain $x = -1 \pm \sqrt{-1}$.

Hence the five required roots are -2 , $+3$, -3 , $-1 + i$, $-1 - i$.

514. Formation of equations. If all roots of an equation are given, the equation can be formed by inspection.

Thus, the equation whose roots are -2 , $+2$, and $+3$ is

$$(x + 2)(x - 2)(x - 3) = 0.$$

Or

$$x^3 - 3x^2 - 4x + 12 = 0.$$

EXERCISE 159

Solve the following equations:

1. $x^3 + 6x^2 + 10x + 8 = 0$, one root being -4 .
2. $x^3 + 7x^2 + 7x - 15 = 0$, one root being -3 .
3. $x^3 - 5x^2 - 9x + 45 = 0$, one root being 5 .
4. $x^3 + x^2 - x - 10 = 0$, one root being 2 .
5. $2x^3 + 7x^2 + 2x - 3 = 0$, one root being -1 .
6. $3x^3 - 16x^2 + 23x - 6 = 0$, one root being 2 .
7. $2x^3 + 3x^2 - 29x + 30 = 0$, one root being -5 .
8. $x^4 - 2x^2 - 3x - 2 = 0$, two roots being -1 , 2 .
9. $3x^4 - 16x^3 + 14x^2 + 24x - 9 = 0$, two roots being -1 , 3 .

Solve by trial:

10. $x^3 - 7x^2 + 16x - 12 = 0$.
11. $x^3 - 12x^2 + 47x - 60 = 0$.
12. $x^3 - 10x^2 + 31x - 30 = 0$.

$$13. \quad x^3 - 8x^2 + 19x - 12 = 0.$$

$$14. \quad x^3 - 6x^2 + 11x - 6 = 0.$$

$$15. \quad x^3 + 9x^2 + 27x + 27 = 0.$$

$$16. \quad x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$$

$$17. \quad 2x^4 - 3x^3 - 12x^2 + 7x + 6 = 0.$$

Form the equations whose roots are :

$$18. \quad 1, -2, 3, -4.$$

$$22. \quad \pm\sqrt{5}, \pm 6.$$

$$19. \quad 1, 2, 3, 0.$$

$$23. \quad 3, 2, 1 + \sqrt{2}, 1 - \sqrt{2}.$$

$$20. \quad 2 + \sqrt{5}, 2 - \sqrt{5}, 3.$$

$$24. \quad -1, +1, 1 + i, 1 - i.$$

$$21. \quad \pm 2, \pm\sqrt{5}.$$

$$25. \quad \pm\sqrt{2}, 2 \pm \sqrt{2}.$$

$$26. \quad \text{How many roots has the equation } x^5 = 1?$$

$$27. \quad \text{How many roots has the equation } \sqrt[7]{x} = 1?$$

RELATIONS BETWEEN THE ROOTS AND THE COEFFICIENTS

515. The equation whose roots are r_1, r_2, r_3 is

$$(x - r_1)(x - r_2)(x - r_3) = 0.$$

Or, expanding,

$$x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 = 0.$$

Comparing this result with the general cubic equation in its simplest form, viz. :

$$x^3 + p_1x^2 + p_2x + p_3 = 0, \text{ we obtain}$$

$$r_1 + r_2 + r_3 = -p_1.$$

$$r_1r_2 + r_1r_3 + r_2r_3 = p_2.$$

$$r_1r_2r_3 = -p_3.$$

516. In a similar manner we obtain for n roots,

$$(x - r_1)(x - r_2)(x - r_3) \cdots (x - r_n) = 0.$$

Or, expanding,

$$\begin{aligned} x^n - (r_1 + r_2 + \cdots r_n)x^{n-1} + (r_1r_2 + r_1r_3 + r_2r_3 + r_1r_4 + \cdots)x^{n-2} \\ - (r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + \cdots)x^{n-3} + \cdots (-1)^n r_1r_2r_3 \cdots r_n = 0. \end{aligned}$$

The general equation for the n th degree in its simplest form is

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0.$$

Comparing the coefficients, we have

$$r_1 + r_2 + r_3 \cdots + r_n = -p_1.$$

$$r_1r_2 + r_1r_3 + r_2r_3 + \cdots = p_2.$$

$$r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + \cdots = -p_3.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$r_1r_2r_3 \cdots r_n = (-1)^n p_n.$$

517. *I.e.* in an equation in its simplest form :

1. *The sum of the roots is equal to the coefficient of the second term with its sign changed.*

2. *The sum of the products of the roots taken two at a time is equal to the coefficient of the third term.*

3. *The sum of the products of the roots taken three at a time is equal to the coefficient of the fourth term with its sign changed, etc.*

4. *The product of all roots is equal to the last term, and is positive or negative according as the degree of the equation is even or odd.*

E.g. if $x^3 + 2x + 5 = 0$, the three roots r_1, r_2, r_3 , satisfy the following equations :

$$r_1 + r_2 + r_3 = 0.$$

$$r_1r_2 + r_1r_3 + r_2r_3 = 2.$$

$$r_1r_2r_3 = -5.$$

518. The following special cases should be noted:

1. If the second term is wanting, the sum of the roots is zero.
2. If the absolute term is wanting, at least one root is zero.

519. The relations between the roots and the coefficients cannot be used to solve the equation, as the work always leads to the original equation.

E.g. let
$$x^3 - 3x^2 + 2x + 5 = 0.$$

Then
$$r_1 + r_2 + r_3 = 3.$$

$$r_1 r_2 + r_1 r_3 + r_2 r_3 = 2.$$

$$r_1 r_2 r_3 = -5.$$

Eliminating r_2 and r_3 ,
$$r_1^3 - 3r_1^2 + 2r_1 + 5 = 0.$$

520. If, however, certain relations exist *between the roots*, the equation can frequently be solved by means of the preceding propositions.

Ex. 1. Solve $4x^3 - 12x^2 + 11x - 3 = 0$, if the roots are in arithmetical progression.

Let $r - s$, r , $r + s$, be the three roots.

Reducing the equation to its simplest form,

$$x^3 - 3x^2 + \frac{11x}{4} - \frac{3}{4} = 0.$$

Therefore
$$3r = 3. \tag{1}$$

$$3r^2 - s^2 = \frac{11}{4}. \tag{2}$$

$$r(r^2 - s^2) = \frac{3}{4}. \tag{3}$$

Solving (1) and (2),
$$r = 1, \quad s = \frac{1}{2}.$$

Since these values also satisfy equation (3), the required roots are $\frac{1}{2}$, 1, $\frac{3}{2}$.

Ex. 2. Determine n so that one root of $x^3 - 7x^2 + nx - 8 = 0$ is the double of another root, and solve the equation.

Let r_1 , r_2 , and $2r_1$ be the three roots.

$$\text{Then} \qquad 3r_1 + r_2 = 7. \qquad (1)$$

$$3r_1r_2 + 2r_1^2 = n. \qquad (2)$$

$$r_1^2r_2 = 4. \qquad (3)$$

$$\text{From (1)} \qquad r_2 = 7 - 3r_1.$$

Substituting in (3) and simplifying,

$$3r_1^3 - 7r_1^2 + 4 = 0.$$

$$\text{Solving by trial,} \qquad r_1 = 1, 2, \text{ or } -\frac{2}{3}.$$

$$\text{Hence} \qquad r_2 = 4, 1, \text{ or } 9;$$

$$2r_1 = 2, 4, \text{ or } -\frac{4}{3};$$

$$\text{and} \qquad n = 14, 14, \text{ or } -17\frac{1}{3}.$$

Ex. 3. In the equation $x^3 + ax^2 + bx - c = 0$, find the condition that the sum of two roots is zero.

Let r_1 , r_2 , and $-r_1$ be the three roots.

$$\text{Then} \qquad r_2 = -a, \quad r_1^2 = -b, \quad r_1^2r_2 = c.$$

$$\text{Eliminating } r_1, r_2, \qquad ab = c.$$

521. A function is **symmetrical** with respect to two letters if an interchange of the two letters does not change the function.

$a^2 + b^2$, $x^3 + 3xy + y^3$, are symmetrical functions.

522. A function is **symmetrical** with respect to three or more letters, if an interchange of any two of these letters does not change the function.

abc , $a^2 + b^2 + c^2$, $a + b + c + a^2b^2c^2$, are symmetrical.

523. Symmetrical functions of the roots of an equation can be found without solving the equation.

Ex. 4. If r_1, r_2, r_3 , are the roots of $x^3 + 3x^2 - 2x + 5 = 0$, find $r_1^2 + r_2^2 + r_3^2$.

$$\begin{aligned} r_1^2 + r_2^2 + r_3^2 &= (r_1 + r_2 + r_3)^2 - 2(r_1r_2 + r_1r_3 + r_2r_3) \\ &= (-3)^2 - 2(-2). \end{aligned}$$

Or, $r_1^2 + r_2^2 + r_3^2 = 13$.

EXERCISE 160

Solve the following equations:

1. $x^3 - 2x^2 - 9x + 18 = 0$, the sum of two roots being zero.
2. $x^3 - 5x^2 + 3x + 9 = 0$, two roots being equal.
3. $4x^3 - 32x^2 - x + 8 = 0$, the sum of two roots being zero.
4. $x^3 - 3x^2 - 13x + 15 = 0$, the roots being in A.P.
5. $x^3 + x^2 - 10x + 8 = 0$, one root being twice another.
6. $x^3 - 2x^2 - 5x + 6 = 0$, one root being three times another.
7. $x^3 - 8x^2 + 19x - 12 = 0$, one root being equal to the sum of the other two.
8. $x^3 - 3x^2 - 6x + 8 = 0$, one root being one-half the sum of the other two.
9. $x^3 - 9x^2 + 26x - 24 = 0$, the roots being in A.P.
10. $x^3 + 7x^2 + 14x + 8 = 0$, the roots being in G.P.
11. $x^3 - 14x^2 + 56x - 64 = 0$, the roots being in G.P.
12. $x^3 + 5x^2 - 4x - 20 = 0$, the sum of two roots being zero.
13. Determine n so that one root of the equation $x^3 - 3x^2 - 10x + n = 0$ is the double of another, and solve the equation.
14. Determine n so that the sum of two roots of the equation $x^3 - 5x^2 - 4x + n = 0$, is equal to zero.
15. The equation $x^3 - ax^2 + bx - c = 0$ has two roots whose sum is zero. Find (a) the third root, (b) the other two roots, (c) a relation between the three coefficients.

If r_1, r_2, r_3 , are the roots of the equation $x^3 + ax^2 + bx + c = 0$, find without solving:

$$16. \quad r_1 + r_2 + r_3 - 3r_1r_2r_3.$$

$$17. \quad (r_1 + r_2 + r_3)^2 - 4(r_1r_2 + r_1r_3 + r_2r_3).$$

$$18. \quad r_1^2 + r_2^2 + r_3^2.$$

$$19. \quad (r_1 + r_2)^2 + (r_2 + r_3)^2 + (r_3 + r_1)^2.$$

20. In the equation $x^3 - 2x^2 - 9x + 1 = 0$, find the sum of the squares of the roots.

21. If one root of the equation $x^3 + ax^2 + bx + c = 0$ is the reciprocal of another one, find the third root.

INCOMMENSURABLE AND IMAGINARY ROOTS

524. *If all coefficients of a rational, integral equation are integers, and the first coefficient is unity, the roots cannot be fractional.*

Let $\frac{r}{s}$ be a root of the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0, \quad (1)$$

and suppose that r and s have no common factor.

Substituting,

$$\left(\frac{r}{s}\right)^n + p_1\left(\frac{r}{s}\right)^{n-1} + p_2\left(\frac{r}{s}\right)^{n-2} + \dots + p_n = 0.$$

Multiplying by s^{n-1} , and transposing,

$$\frac{r^n}{s} = -p_1r^{n-1} - p_2r^{n-2}s - p_3r^{n-3}s^2 - \dots - p_ns^{n-1},$$

which is impossible, as the left member is a fraction in its lowest terms, and the right member is an integer.

Hence equation (1) cannot have fractional roots.

525. *If the coefficients of a rational, integral equation are integers, all integral roots are factors of the absolute term.*

Suppose r is an integral root of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0.$$

Substituting, transposing, and dividing by r ,

$$a_0r^{n-1} + a_1r^{n-2} + a_2r^{n-3} + \cdots + a_{n-1} = -\frac{a_n}{r}.$$

Since the left number is an integer, the right member must also be an integer; *i.e.* r must be a factor of a_n .

526. *If a complex number is a root of an equation with real coefficients, $f(x) = 0$, its conjugate is also a root.*

Let $a + bi$ be a root, and $Q = f(x) \div [x - (a + bi)]$.

$$\text{Then} \quad [x - (a + bi)]Q = f(x). \quad (1)$$

As Q involves i , we would obtain a different expression, Q' , if we would substitute $-i$ in place of i .

Since the product of $x - (a + bi)$ and Q (*i.e.* $f(x)$) is real, it contains only even powers of i .

Hence, if we substitute $-i$ in place of i in the left member of (1), the product must be the same as before, as the even powers of i and $-i$ are identical.

$$\text{Hence} \quad [x - (a - bi)]Q' = f(x).$$

I.e. $a - bi$ is a root of $f(x) = 0$.

527. *To every pair of imaginary roots there corresponds the quadratic factor $(x - a)^2 + b^2$, which is real and positive for any real value of x .*

Hence any rational, integral function of x can be resolved into real factors, which are either linear or quadratic in x .

528. If $a + \sqrt{b}$ is a root of an equation whose coefficients are rational, $a - \sqrt{b}$ is also a root.

The proof is similar to that of § 526.

Ex. 1. Solve the equation

$$x^4 - 5x^3 + 5x^2 + 17x - 42 = 0,$$

one root being $2 - \sqrt{-3}$.

Since $2 - \sqrt{-3}$ is a root, $2 + \sqrt{-3}$ is also a root.

The quadratic factor corresponding to these roots is

$$(x - 2 + \sqrt{-3})(x - 2 - \sqrt{-3}), \text{ or } x^2 - 4x + 7.$$

Removing this factor by division, we obtain the depressed equation $x^2 - x - 6 = 0$, whose roots are -2 and 3 .

Hence the four roots are $2 - \sqrt{-3}$, $2 + \sqrt{-3}$, -2 , 3 .

Ex. 2. Form an equation with rational coefficients, one of whose roots is $\sqrt{2} + \sqrt{3}$.

Since $\sqrt{2} + \sqrt{3}$ is a root, $\sqrt{2} - \sqrt{3}$ must be a root.

Similarly, $-\sqrt{2} + \sqrt{3}$ and $-\sqrt{2} - \sqrt{3}$ are roots.

Hence the required equation is

$$(x - \sqrt{2} - \sqrt{3})(x - \sqrt{2} + \sqrt{3})(x + \sqrt{2} - \sqrt{3})(x + \sqrt{2} + \sqrt{3}) = 0.$$

Or

$$x^4 - 10x^2 + 1 = 0.$$

EXERCISE 161

Solve the following equations, having given the indicated root:

- | | |
|--------------------------------------|--------------------|
| 1. $x^3 - 2x - 4 = 0;$ | $-1 + \sqrt{-1}.$ |
| 2. $x^3 + 48x + 504 = 0;$ | $3 + 5\sqrt{-3}.$ |
| 3. $x^3 - 21x - 344 = 0;$ | $-4 + 3\sqrt{-3}.$ |
| 4. $x^3 - 9x + 28 = 0;$ | $2 + \sqrt{-3}.$ |
| 5. $x^4 - 3x^3 + 3x^2 - 3x + 2 = 0;$ | $i.$ |

$$6. \quad x^4 - 7x^2 - 12x + 18 = 0; \quad -2 + \sqrt{-2}.$$

$$7. \quad x^4 - 10x^3 + 33x^2 - 46x + 20 = 0; \quad 3 + \sqrt{5}.$$

$$8. \quad x^4 - 8x^3 + 14x^2 + 4x - 8 = 0; \quad 1 + \sqrt{3}.$$

$$9. \quad x(x+1)(x+2)(x+3) = 24; \quad -\frac{3}{2} + \frac{1}{2}\sqrt{-15}.$$

10. If $-2 + \sqrt{-7}$ is a root of $x^4 + 2x^2 - 16x + 77 = 0$, resolve the left member into real factors.

11. If $2 + i$ is a root of $x^3 - 3x^2 + x + 5 = 0$, resolve the left member into real factors.

12. Can 7 be a root of the equation $x^5 + 2x^2 + 5x + 12 = 0$? Can $\frac{1}{2}$ be a root of the same equation?

13. Form an equation with rational coefficients, one of whose roots is $\sqrt{2} + i$.

14. Form an equation with rational coefficients, one of whose roots is $\sqrt{3} - 2i$.

TRANSFORMATION OF EQUATIONS

529. The solution of an equation is sometimes simplified by transforming it into another one whose roots bear a certain relation to the original one.

530. The method used for most transformations consists in expressing the relation between x and the new variable y in the form of an equation. Find x in terms of y , and substitute this value in the given equation.

Ex. 1. Find an equation whose roots are the reciprocals of the roots of the equation $2x^4 + 3x^3 + 4x^2 + 5x + 6 = 0$.

$$\text{Make } y = \frac{1}{x}, \text{ then} \quad x = \frac{1}{y}.$$

$$\text{Substituting,} \quad \frac{2}{y^4} + \frac{3}{y^3} + \frac{4}{y^2} + \frac{5}{y} + 6 = 0.$$

$$\text{Multiplying by } y^4, \quad 6y^4 + 5y^3 + 4y^2 + 3y + 2 = 0.$$

Ex. 2. Find an equation whose roots are the cubes of the roots of the equation $f(x) = 0$.

Make $y = x^3$, then $x = \sqrt[3]{y}$, therefore $f(\sqrt[3]{y})$ is the required equation.

Ex. 3. Find an equation the cubes of whose roots are seven times the squares of the roots of the equation $f(x) = 0$.

Let $y^3 = 7x^2$, then $x = \sqrt{\frac{y^3}{7}}$, therefore $f\left(\sqrt{\frac{y^3}{7}}\right) = 0$.

531. Transform the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$$

into another whose roots are those of the given equation with their signs changed.

Let $y = -x$, then $x = -y$.

Substituting,

$$(-y)^n + p_1(-y)^{n-1} + p_2(-y)^{n-2} + \cdots + p_{n-1}(-y) + p_n = 0.$$

Simplifying, and dividing by -1 if n should be odd,

$$y^n - p_1y^{n-1} + p_2y^{n-2} - p_3y^{n-3} + \cdots (-1)^np_n = 0.$$

E.g. to change the signs of the roots of the equation

$$x^5 - 4x^4 + 3x^3 + 2x - 5 = 0,$$

change the signs of all even powers,

i.e.
$$x^5 + 4x^4 + 3x^3 + 2x + 5 = 0.$$

532. To transform the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0$$

into another one whose roots are equal to m times the roots of the given equation.

Let $y = mx$, then $x = \frac{y}{m}$.

Substituting,

$$\left(\frac{y}{m}\right)^n + p_1\left(\frac{y}{m}\right)^{n-1} + p_2\left(\frac{y}{m}\right)^{n-2} + \cdots + p_n = 0.$$

Multiplying by m^n ,

$$y^n + p_1 y^{n-1} m + p_2 y^{n-2} m^2 + \cdots + p_n m^n = 0.$$

I.e. the required equation is obtained by multiplying the second term by m , the third by m^2 , etc.

Thus, the equation whose roots are ten times the roots of

$$x^3 - 4x^2 + 2x - 7 = 0,$$

is

$$x^3 - 40x^2 + 200x - 7000 = 0.$$

533. The principal application of the preceding transformation consists in clearing equations of fractions.

Ex. 1. Transform the equation

$$x^3 - \frac{5}{2}x^2 + \frac{7}{36}x - \frac{5}{72} = 0$$

into another one with integral coefficients.

Substituting $x = \frac{y}{m}$,

$$y^3 - \frac{5}{2}y^2m + \frac{7}{36}ym^2 - \frac{5}{72}m^3 = 0.$$

Let $m = 6$, then

$$y^3 - 15y^2 + 7y - 15 = 0.$$

Ex. 2. Transform the equation

$$16x^3 - 4x^2 + 2x + 3 = 0$$

into another one which has integral coefficients and unity for the coefficient of x^3 .

Dividing by 16,

$$x^3 - \frac{1}{4}x^2 + \frac{1}{8}x + \frac{3}{16} = 0.$$

Let $x = \frac{y}{m}$,

$$y^3 - \frac{1}{4}y^2m + \frac{1}{8}ym^2 + \frac{3}{16}m^3 = 0.$$

Let $m = 4$, *i.e.* $x = \frac{y}{4}$.

Then

$$y^3 - y^2 + 2y + 12 = 0.$$

Ex. 3. Solve the equation

$$x^3 - \frac{7}{2}x^2 + 4x - \frac{3}{2} = 0. \quad (1)$$

$$\text{Let } x = \frac{y}{2}, \quad y^3 - 7y^2 + 16y - 12 = 0. \quad (2)$$

If there are any rational roots, they must be factors of 12, *i.e.* ± 1 , ± 2 , ± 3 , etc.

As ± 1 evidently does not satisfy (2), try $y = 2$.

$$\begin{array}{r} 1 - 7 + 16 - 12 \quad | 2 \\ 2 - 10 + 12 \\ \hline 1 - 5 + 6 \quad 0 \end{array}$$

$$\text{Hence} \quad (y - 2)(y^2 - 5y + 6) = 0.$$

$$\text{Or} \quad (y - 2)(y - 2)(y - 3) = 0.$$

$$\text{I.e.} \quad y = 2, 2, \text{ or } 3.$$

$$\text{Hence} \quad x = 1, 1, \text{ or } \frac{3}{2}.$$

EXERCISE 162

1. Transform the equation $x^5 - 7x^4 + 2x^3 + 2x^2 - 3x - 7 = 0$ into another one whose roots are those of the given equation with signs changed.

2. Transform the equation $x^3 - 7x^2 + 2x - 9 = 0$ into another one whose roots are five times the roots of the original equation.

3. Transform $x^6 - 2x^5 + 3x^4 - 4x^3 + 5x^2 - 6x + 7 = 0$ so that its roots equal ten times the original roots.

Transform the following equations so that the first coefficient is unity, and all coefficients are integers:

$$4. \quad x^3 - \frac{4}{3}x^2 + \frac{4}{3}x - \frac{1}{9} = 0. \quad 6. \quad x^3 - \frac{4}{3}x^2 - \frac{4}{12}x + \frac{1}{72} = 0.$$

$$5. \quad x^3 - \frac{5}{2}x^2 + \frac{7}{4}x - \frac{3}{2} = 0. \quad 7. \quad x^3 - \frac{1}{4}x^2 - \frac{1}{8}x - \frac{1}{32} = 0.$$

$$8. \quad x^4 - 3x^3 - \frac{5}{2}x^2 + \frac{5}{4}x + \frac{1}{16} = 0.$$

$$9. \quad 2x^4 - 4x^3 - 3x^2 + 2x - \frac{1}{2} = 0.$$

$$10. \quad 3x^3 - 6x^2 + 4x - \frac{1}{3} = 0.$$

Solve by trial:

$$11. x^3 - 3x^2 + \frac{1}{4}x - \frac{3}{4} = 0.$$

$$14. 9x^3 - 12x^2 + x + 2 = 0.$$

$$12. x^3 + \frac{8}{5}x^2 + \frac{1}{5}x - \frac{2}{5} = 0.$$

$$15. 8x^3 - 12x^2 - 4x + 6 = 0.$$

$$13. x^3 + \frac{2}{3}x^2 - \frac{2}{9}x + \frac{2}{9} = 0.$$

$$16. x^3 - 2x^2 + \frac{1}{4}x - \frac{1}{2} = 0.$$

DESCARTES' RULE OF SIGNS

534. In a series of algebraic numbers, a **permanence** is the succession of two like terms; a **variation** is the succession of two unlike terms.

Thus, $++--$ contains two permanences and one variation, and $+-+--+-+$ contains one permanence and six variations.

535. Descartes' Rule of Signs. *The number of positive roots in an equation cannot exceed the number of variations, and the number of negative roots cannot exceed the number of permanences.*

To prove the theorem for positive roots, it is only necessary to show that for the introduction of each positive root, at least one variation is added.

Suppose the signs of a complete equation to be

$$++-+- - - + -.$$

To introduce a positive root a , we have to multiply by $x - a$. Writing only the signs, we have

$$\begin{array}{r} ++-+- - - + - \\ - - + - + + + - + \\ \hline + \pm - + - \pm \pm + - + \end{array}$$

In the result the ambiguous sign \pm is used wherever the sign is doubtful.

Comparing the signs of the original expression with the signs of the product, we observe the following facts, which can easily be proved:

1. The sign of the first term is not changed.
2. The second sign of any variation is not changed.
3. The second sign of any permanence becomes ambiguous.
4. There is added one term at the end whose sign is opposite to the preceding one.

Hence the only signs which may be changed are the second terms of the permanences. But a change in the last term of the permanences cannot decrease the number of variations.

Considering the additional variation at the end, it is evident that the product must contain at least one variation more than the original expression.

Hence the total number of positive roots cannot be greater than the number of variations.

536. To prove the theorem for negative roots, consider the equation

$$f(x) = 0, \quad (1)$$

and change the signs of all roots of (1) by substituting $-x$ for x .

$$\text{I.e.} \quad f(-x) = 0. \quad (2)$$

Since (2) is obtained from (1) by changing the signs of the even terms, the number of permanences in (1) equals the number of variations in (2).

But (2) cannot have more positive roots than it has variations. Hence (1) cannot have more negative roots than it has permanences.

E.g. $4x^5 - 2x^4 - 6x^3 + 7x^2 + 2x - 1$ cannot have more than three positive and two negative roots.

537. Incomplete equations. The preceding proof refers only to complete equations, but every incomplete equation can be completed by the introduction of zero coefficients.

Thus, $x^6 - 4x^3 - 3x - 7 = 0$ may be written

$$x^6 \pm 0x^5 \pm 0x^4 - 4x^3 \pm 0x^2 - 3x - 7 = 0.$$

The smallest number of variations is obtained by making the signs of the zero coefficients agree with the preceding terms, viz.

$$+ 1 + 0 + 0 - 4 - 0 - 3 - 7.$$

Evidently the number of variations is then the same as in the original equation, and hence *Descartes' Rule for positive roots may be applied to incomplete equations.*

538. For negative roots, however, the number of permanences can often be made smaller by the introduction of zero coefficients. *E.g.* the above equation has two permanences. But writing the coefficients as follows:

$$1 - 0 + 0 - 4 + 0 - 3 - 7,$$

the equation has only one permanence.

Hence *Descartes' Rule for negative roots should not be applied directly to incomplete equations.* The most convenient method in such a case is to transform the equation into another one whose roots have opposite signs, as illustrated in the next article.

539. In many incomplete equations, the greatest possible number of real roots does not equal the degree of the equation, hence in many cases imaginary roots can be detected by the rule.

Ex. Prove that $x^6 + 3x^2 - 5x + 1 = 0$ has at least four imaginary roots.

As the equation has two variations, it cannot have more than two positive roots.

Substituting $-x$ for x ,

$$x^6 + 3x^2 + 5x + 1 = 0.$$

Since this equation has no variations, it cannot have any positive root, and hence the original equation cannot have any negative roots. That is, the total number of real roots cannot exceed two.

Therefore the given equation has at least four imaginary roots.

540. It follows from Descartes' Rule that :

1. *If all signs of an equation are positive, the equation has no positive roots.*

2. *If the signs of a complete equation are alternately positive and negative, the equation has no negative roots.*

EXERCISE 163

Apply Descartes' Rule to the following equations :

1. $x^5 - 2x^4 - 3x^3 - 4x^2 - 2x + 1 = 0.$

2. $x^6 - 5x^5 + 2x^4 - 3x^3 + 2x^2 - 3x - 7 = 0.$

3. $5x^6 - 6x^5 + 2x^4 + 3x^3 + 5x^2 + 2x + 1 = 0.$

4. $6x^6 + x^5 + 7x^4 + 9x^3 - 2x^2 - 5x - 6 = 0.$

All the roots of the following equations are real; determine their signs :

5. $8x^3 + 12x^2 - 4x - 1 = 0.$

6. $27x^3 - 54x^2 + 25x + 1 = 0.$

7. $2x^3 - 5x^2 - 13x + 30 = 0.$

8. $x^4 - 2x^3 - 11x^2 + 6x + 2 = 0.$

9. $4x^4 - 4x^3 - 13x^2 + 18x - 6 = 0.$

10. $4x^5 - 41x^4 + 6x^2 + 73x + 30 = 0.$

Determine the least possible number of imaginary roots in each of the following equations :

11. $x^5 + 2x^2 + x - 2 = 0.$

12. $x^6 - 3x^3 + 2x - 9 = 0.$

13. $x^7 + 2x^5 - 7x^2 - 9 = 0.$

14. $x^6 + 2x^5 - 2x - 7 = 0.$

15. $x^6 + 2x^5 - 3x^4 + 2x + 1 = 0$.
16. $3x^7 - 5x^4 + 4x^3 - 5 = 0$.
17. $x^{10} - 1 = 0$.
18. $x^{93} - 1 = 0$.
19. $x^n + 1 = 0$, if n is even.
20. $x^6 + 2x^4 + 2x^2 + 5 = 0$.
21. $ax^3 + bx + c = 0$, if a , b , and c are positive.

LOCATION OF ROOTS

541. Limits of roots. The solution of numerical equations may often be simplified by determining between what limits the roots lie.

542. A **superior limit** to the real roots is a number greater than the greatest root.

543. An **inferior limit** to the real roots is a number smaller than the smallest root.

544. A **superior limit** may be obtained by grouping the terms so that each group contains not more than one negative term, and determining a value of x which makes each group positive.

Ex. 1. Find a superior limit to the roots of the equation

$$x^4 + 4x^3 - 6x^2 + 24x - 60 = 0.$$

Grouping terms and factoring,

$$x^2(x^2 - 6) + 4(x^3 - 15) + 24x = 0.$$

Evidently each term is positive, if $x \geq 2\frac{1}{2}$.

Hence $2\frac{1}{2}$ is a superior limit.

Ex. 2. Find a superior limit to the roots of the equation

$$x^3 - 2x^2 - 3x - 16 = 0.$$

Multiplying by 3, $3x^3 - 6x^2 - 9x - 48 = 0.$

Grouping, $(x^3 - 6x^2) + (x^3 - 9x) + (x^3 - 48) = 0.$

Or $x^2(x - 6) + x(x^2 - 9) + (x^3 - 48) = 0.$

Evidently no term is negative if $x \geq 6.$

Hence 6 is a superior limit.

Since the limits obtained from the three groups differ considerably, we modify the method as follows :

Multiplying the given equation by 4, and grouping,

$$(2x^3 - 8x^2) + (x^3 - 12x) + (x^3 - 64) = 0.$$

Factoring, $2x^2(x - 4) + x(x^2 - 12) + (x^3 - 64) = 0.$

No term is negative, if $x \geq 4.$

Hence 4 is a superior limit.

545. To determine an inferior limit, change the signs of all roots by the transformation of § 531, and determine the superior limit.

Ex. 3. Find an inferior limit to the roots of the equation

$$x^4 - 20x^2 + 48x + 120 = 0.$$

Let $x = -y.$

Then $y^4 - 20y^2 - 48y + 120 = 0$ (§ 531). (1)

Multiplying by 2, and grouping, $(y^4 - 40y^2) + (y^4 - 96y) + 240 = 0.$

Factoring, $y^2(y^2 - 40) + y(y^3 - 96) + 240 = 0.$

I.e. $6\frac{1}{2}$ is a superior limit to $y.$

Therefore $-6\frac{1}{2}$ is an inferior limit to $x.$

A closer limit is obtained by multiplying (1) by 3.

$$2y^4 - 60y^2 + y^4 - 144y + 360 = 0.$$

$$2y^2(y^2 - 30) + y(y^3 - 144) + 360 = 0.$$

I.e. $5\frac{1}{2}$ is a superior limit to $y.$

Therefore $-5\frac{1}{2}$ is an inferior limit to $x.$

EXERCISE 164

Determine a superior and an inferior limit to the roots of the following equations :

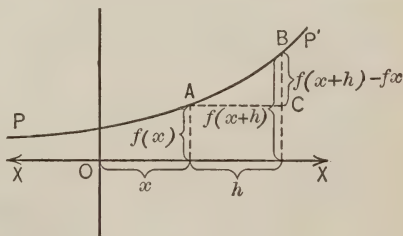
1. $x^3 - 3x^2 + 5x - 12 = 0$.
2. $x^4 + 3x^2 - 4x - 14 = 0$.
3. $3x^3 - 5x^2 + 3x - 9 = 0$.
4. $x^3 - 3x^2 - 2x + 1 = 0$.
5. $2x^4 - x^3 - 2x^2 + x - 4 = 0$.
6. $3x^3 - 4x^2 - 5x - 7 = 0$.
7. $4x^4 + 2x^3 + 2x^2 - 7x + 24 = 0$.
8. $5x^5 + 2x^4 - 3x^3 + 2x^2 - x - 48 = 0$.
9. $6x^6 + 5x^5 + 4x^3 + 3x^2 + 2x + 1 = 0$.
10. $x^6 - 2x^2 - 12600 = 0$.

546. A function $f(x)$ is **continuous** if, for any value of x , an infinitesimal change in the independent variable x produces an infinitesimal change in the function, or if

$$f(x+h) - f(x) \doteq 0, \text{ if } h \doteq 0.$$

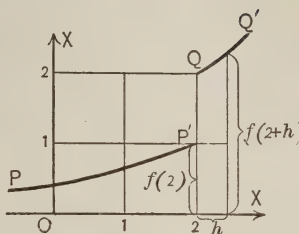
Thus, x^2 is continuous for any value of x , as $(x+h)^2 - x^2 = 2hx + h^2$, a quantity which for any finite x approaches zero as a limit, if $h \doteq 0$.

547. The meaning of continuity is well illustrated by representing functions graphically. The line PP' , which represents the graph of a function $f(x)$, is supposed to be a continuous line, *i.e.* a line without a break. Such a function is a continuous function, and the inspection of the diagram shows that any increase (h or AC) in the independent variable and the corresponding increase (cB) of the function simultaneously approach zero as a limit.



548. On the other hand, the second diagram represents a discontinuous function ($PP'QQ'$). It is apparent that $f(2)$ will have the finite increase of $+1$ if 2 increases by an infinitesimal quantity h .

Hence this function is discontinuous.



549. The function ax^m is continuous if m is an integer.

For $a(x+h)^m - ax^m$

$$= a[x^m + mx^{m-1}h + {}^mC_2x^{m-2}h^2 + \dots - x^m]$$

$$= a \cdot h[mx^{m-1} + {}^mC_2x^{m-2}h + \dots].$$

If $h \doteq 0$, the right member $\doteq 0$, for any finite x .

Hence ax^m is continuous

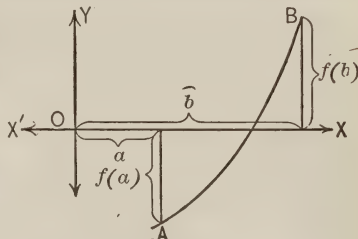
550. The function $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ is continuous.

Any infinitesimal increase of h produces an infinitesimal increase of each term, and therefore of the whole function (§ 412, 1).

Hence $f(x)$ is continuous.

551. If $f(a)$ and $f(b)$ have opposite signs, at least one root must lie between a and b .

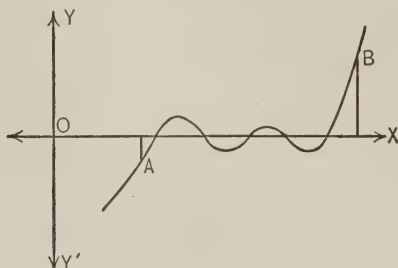
If x changes gradually from a to b , $f(x)$ changes gradually from $f(a)$ to $f(b)$, i.e. from a positive to a negative, or from a negative to a positive, value. Hence it must pass at least once through zero, i.e. $f(x)$ has at least one root between a and b .



552. The preceding proposition becomes obvious by inspection of the graph. Evidently a

continuous line cannot join two points, A and B , on opposite sides of OX without intersecting OX at least once.

There may, however, be three, five, or any odd number of roots, as appears from the next diagram.



553. The preceding article furnishes the principal method for locating incommensurable roots.

Ex. 1. Show that the equation

$$x^4 + 5x^3 - 60x^2 + 70x + 100 = 0$$

has a root between 4 and 5.

$$\begin{array}{r} 1 + 5 - 60 + 70 + 100 \quad \underline{4} \\ + 4 + 36 - 96 - 104 \\ \hline 1 + 9 - 24 - 26 ; - 4. \quad \text{I.e. } f(4) = -4. \end{array}$$

$$\begin{array}{r} 1 + 5 - 60 + 70 + 100 \quad \underline{5} \\ + 5 + 50 - 50 + 100 \\ \hline 1 + 10 - 10 + 20 ; + 200. \quad \text{I.e. } f(5) = +200. \end{array}$$

Hence at least one root lies between 4 and 5.

554. If $x \doteq \infty$, then $f(x) \doteq +\infty$.

If $x \doteq -\infty$, then $f(x) \doteq +\infty$ if n is even, and $f(x) \doteq -\infty$ if n is odd.

For it can be shown that, for a very large value of x , the first term of the function is greater than the sum of the remaining terms, and hence the function approaches the values $+\infty$ or $-\infty$.

555. *Every equation of an odd degree has at least one real root whose sign is opposite to that of the last term.*

If x equals respectively $-\infty, 0, +\infty$,
then $f(x)$ equals respectively $-\infty, p_n, +\infty$.

Hence if p_n is positive, there must be at least one root between $-\infty$ and 0, *i.e.* there must be a negative root.

Similarly if p_n is negative, there must be a positive root.

Thus, $x^3 + x^2 + x + 1$ has at least one negative real root.

556. *Every equation of an even degree and a negative absolute term (p_n) has at least one positive and one negative root.*

If x equals respectively $-\infty, 0, +\infty$,
 $f(x)$ equals respectively $+\infty, -, +\infty$.

Hence there must be at least one positive and one negative root.

Thus, $x^3 + 4x^2 - 2x - 7 = 0$ has *at least* one positive and one negative root.

Ex. 2. Locate the real roots of the equation

$$x^4 - 10x^3 + 33x^2 - 40x + 14 = 0.$$

Since there are no permanences, there can be no negative roots, and the method of § 544 shows that 10 is a superior limit.

Evidently $f(0) = 14$, and $f(1) = -2$. By synthetic division we obtain $f(2) = 2$, $f(3) = 2$, $f(4) = -2$, and $f(5) = +14$.

Hence the four roots are incommensurable and lie respectively between 0 and 1, between 1 and 2, between 3 and 4, and between 4 and 5.

Ex. 3. Find the nature of the roots, and locate the real roots of the equation:

$$x^4 - 2x^2 - 6 = 0.$$

Since the degree of the equation is even, and the absolute term is negative, there must be one positive and one negative root (§ 550). Descartes' Rule shows that there can be no more than one positive and one negative root.

Hence there must be one positive, one negative, and two imaginary roots.

Since $x^2(x^2 - 4) + (x^4 - 12)$ is positive if $x = 2$, we have a superior limit 2; and similarly obtain the inferior limit -2 .

Hence we have to find the sign of $f(-2)$, $f(-1)$, $f(0)$, $f(1)$, and $f(2)$. A simple calculation shows that the required five signs are respectively $+ - - - +$.

Therefore one root lies between -2 and -1 , another between 1 and 2 .

EXERCISE 165

Determine the character of the roots of the following equations:

1. $2x^4 + 22x^2 + 5x - 11 = 0$.

2. $x^7 + 3x^5 - 3x^2 - 4 = 0$.

3. $x^4 - 3x^3 - 12x - 16 = 0$.

4. $x^3 + ax - b = 0$, if a and b are positive.

5. $x^3 + ax + b = 0$, if a and b are positive.

Locate the roots of the following equations:

6. $4x^3 - 10x^2 + 2x + 3 = 0$. 9. $x^3 + 4x - 5 = 0$.

7. $x^3 + 4x^2 - 4x - 8 = 0$. 10. $x^3 + 2x + 2 = 0$.

8. $x^3 - 6x^2 + 4x + 10 = 0$. 11. $x^4 - 5x^3 - 4x + 19 = 0$.

CHAPTER XXIX

SOLUTION OF HIGHER EQUATIONS

COMMENSURABLE ROOTS

557. The principal features of the method for finding commensurable roots were discussed in the preceding chapter. There are, however, a few additional propositions, which greatly facilitate the finding of commensurable roots.

558. *If r is an integral root of the equation with integral coefficients $f(x) = 0$, then $\frac{f(x)}{x - r}$ must be an integer for any integral value of x , for this quotient is a rational, integral function of x .*

Since $f(1)$ is easily found, we usually apply this test first for $x = 1$. *i.e.* $\frac{f(1)}{1 - r}$ must be an integer for any integral root r . Similarly for $\frac{f(-1)}{-1 - r}$, etc.

Ex. 1. Determine the integral roots of the equation

$$10x^4 + 17x^3 - 16x^2 + 2x - 20 = 0.$$

The factors of -20 are $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$.

$f(1) = -7$; that is, 1 is not a root.

If $r = 2, 4, 5, 10, 20, -1, -2, -4, -5, -10, -20$,
then $1 - r = -1, -3, -4, -5, -19, 2, 3, 5, 11, 21$.

Rejecting all values of $1 - r$ which are not factors of -7 , and the corresponding values of r , only 2 remains.

As synthetic division shows that 2 is not a root, the equation has no integral roots

Ex. 2. Determine the integral roots of the equation

$$2x^4 - 49x^3 + 281x^2 + 72x - 180 = 0.$$

The inferior and superior limits are respectively -2 and $+25$. Hence the roots may be $-1, +1, 2, \dots 20$.

$f(1) = 2 - 49 + 281 + 72 - 180 = 126$, hence 1 is not a root.

If $r = -1, 2, 3, 4, 5, 6, 7, 10, 12, 15, 18, 20$,
then $1 - r = 2, -1, -2, -3, -4, -5, -6, -9, -11, -14, -17, -19$.

Rejecting all values of $1 - r$ which are not factors of 126 , and the corresponding values of r , there remain $-1, 2, 3, 4, 10$, and 15 .

$f(-1) = 2 + 49 + 281 - 72 - 180 = 80$; hence -1 is not a root.

If $r = 2, 3, 4, 10, 15$,

$$-1 - r = -3, -4, -5, -11, -16.$$

Rejecting all values of $-1 - r$, which are not factors of 80 , and the corresponding values of r , there remain $3, 4$, and 15 as possible roots.

Synthetic division shows that 3 and 4 are not roots, but 15 is a root. Hence 15 is the only integral root.

559. Newton's method of divisors is very similar to synthetic division; but the numerical work is sometimes simpler, especially when the exponents or the coefficients are very large.

Let r be a root of the equation

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0.$$

Substituting, dividing by r , and transposing,

$$\frac{a_n}{r} + a_{n-1} = -a_{n-2}r - \dots - a_1r^{n-2} - a_0r^{n-1}.$$

The right member being an integer, the left member must be an integer.

Denoting this integer by q_2 , dividing by r , and transposing,

$$\frac{q_2}{r} + a_{n-2} = -a_{n-3}r - \dots - a_1r^{n-3} - a_0r^{n-2}.$$

As before, the left member must be an integer. Denoting it by q_3 , dividing by r , and transposing,

$$\frac{q_3}{r} + a_{n-3} = -a_{n-4}r - \dots - a_1r^{n-4} - a_0r^{n-3}.$$

Continuing this process, we obtain finally

$$\frac{q_{n-1}}{r} + a_1 = -a_0r.$$

$$\frac{q_n}{r} + a_0 = 0.$$

560. The work can be arranged in a manner very similar to the one used for synthetic division. In fact, the work differs from synthetic division only in two points:

- (1) *We commence at the last term and work toward the first.*
- (2) *We divide instead of multiplying.*

Ex. Determine if 2 is a root of

$$2x^4 - x^3 - 10x^2 + 15x - 14 = 0.$$

$$\begin{array}{r} 2 - 1 - 10 + 15 - 14 \quad \underline{2} \\ -2 - 3 + 4 - 7 \\ \hline 0 - 4 - 6 + 8 - 14 \end{array}$$

Explanation. Take down -14 , divide it by 2, and add the quotient (-7) to the preceding coefficient. Divide this sum (8) by 2, and add the quotient to the preceding coefficient, etc. As there is no fractional quotient, and as the last sum is zero, 2 is a root. The coefficients of the depressed equation are the numbers in the second line with their signs changed, viz. $+2 + 3 - 4 + 7$.

561. If a number is not a root, this fact is discovered as soon as we come to a fractional quotient. This constitutes the advantage of Newton's method over synthetic division.

I.e. to determine if 7 is a root of $x^4 + x^3 - 10x^2 + 15x - 14 = 0$, we have

$$\begin{array}{r} 1 + 1 - 10 + 15 - 14 \quad \underline{7} \\ - 2 \\ \hline + 13 - 14 \end{array}$$

Hence 7 is not a root.

562. The method for finding commensurable roots may be summarized as follows:

1. *Make the coefficients integers, and determine all integers that may be roots. To be roots, integers must satisfy the following conditions:*

- (a) *They must be factors of the absolute term.*
- (b) *They must lie between the inferior and the superior limits.*
- (c) *They must conform with Descartes' Rule.*

(d) $\frac{f(1)}{1-r}$ must be an integer, if r is an integral root. Similarly for $\frac{f(-1)}{-1-r}, \frac{f(2)}{2-r}$, etc.

2. *Try if the numbers which satisfy (1) are roots of the equation. The trial may be made by*

- (a) *Synthetic Division.*
- (b) *Newton's Method.*

NOTE. Very small numbers, as $+1$, or -1 , are more easily tested by direct substitution. *E.g.* if $x=1$, then $f(x)$ =the sum of the coefficients.

3. *After all integral roots are removed, the equation has no more commensurable roots, if the first coefficient is unity.*

4. *If the first coefficient is not unity, transform the equation into one in its simplest form (i.e. first coefficient=1) with integral coefficients, and proceed as before.*

NOTE. If the first coefficient is very small, it is sometimes convenient to apply this last transformation (4) at the very beginning.

Ex. 1. Solve the equation

$$2x^4 + 5x^3 - 114x^2 - 238x - 240 = 0.$$

The inferior and superior limits are respectively -11 and 8 .

Hence the roots may be $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, -8, -10$.

$f(1) = 2 + 5 - 114 - 238 - 240 = -585$. Hence 1 is not a root.

If $r = -10, -8, -6, -5, -4, -3, -2, -1, 2, 3, 4, 5, 6$,

$1-r = 11, 9, 7, 6, 5, 4, 3, 2, -1, -2, -3, -4, -5$.

Rejecting all values of $1-r$ which are not factors of -585 , there remain the following values of r , -8 , -4 , -2 , 2 , 4 , 6 .

$$f(-1) = 2 - 5 - 114 + 238 - 240 = -119.$$

If $r = -8, -4, -2, 2, 4, 6$,
then $-1-r = 7, 3, 1, -3, -5, -7$.

Rejecting all values of $-1-r$, which are not factors of -119 , we have $r = -8, -2$, or $+6$.

Testing $x = -2$, *i.e.* dividing by $x+2$, we obtain the remainder -228 ; that is, -2 is not a root, and $f(-2) = -228$.

If $r = -8, +6$,
then $-2-r = 6, -8$.

But -8 is not a factor of -228 , hence the only possible integral root is the one corresponding with 6 , *i.e.* -8 .

Dividing,

$$\begin{array}{r} 2 + .5 - 114 - 238 - 240 \quad | -8 \\ -16 + 88 + 208 + 240 \\ \hline 2 - 11 - 26 - 30 \quad 0 \end{array}$$

I.e. $x = -8$ is the only integral root, and $2x^3 - 11x^2 - 26x - 30 = 0$ is the depressed equation.

Dividing by 2, $x^3 - \frac{11}{2}x^2 - 13x - 15 = 0$.

Let $x = \frac{y}{2}$, $y^3 - 11y^2 - 52y - 120 = 0$.

The limits to y are respectively $-7\frac{1}{2}$ and $+17$.

Rejecting all multiples of 2, since x cannot be an integer, we have the possible roots $\pm 1, \pm 3, \pm 5, +15$.

$$f(1) = 1 - 11 - 52 - 120 = -182. \quad \text{Hence 1 is not a root.}$$

If $r = -5, -3, -1, 3, 5, 15$,
 $1-r = 6, 4, 2, -2, -4, -14$.

I.e. $-1, 3, 15$, are the only integers that may be roots.

$$f(-1) = -1 - 11 + 52 - 120 = -80. \quad \text{Hence } -1 \text{ is not a root.}$$

Synthetic division proves that 3 is not a root, while 15 is a root.

The depressed equation is $y^2 + 4y + 8 = 0$. Solving by formula,

$$y = -2 \pm 2i.$$

Therefore $y = -2 + 2i, -2 - 2i$, or 15 .

Hence the required roots are $-8, \frac{15}{2}, -1+i, -1-i$.

Ex. 2. Solve by Newton's method,

$$8x^4 - 68x^3 + 190x^2 - 199x + 60 = 0.$$

The equation can have positive roots only, and $8\frac{1}{2}$ is a superior limit. Hence we have to try 1, 2, 3, 4, 5, 6.

$$f(1) = 8 - 68 + 190 - 199 + 60 = -9. \quad \text{Hence 1 is not a root.}$$

$$\text{If} \quad r = 2, \quad 3, \quad 4, \quad 5, \quad 6,$$

$$1 - r = -1, \quad -2, \quad -3, \quad -4, \quad -5.$$

Rejecting all values of $1 - r$ which are not factors of -9 , there remain $r = 2$, or 4 .

$$\begin{array}{r} \dots + 190 - 199 + 60 \quad \underline{2} \\ + 30 \\ \hline -169 + 60 \\ \\ 8 - 68 + 190 - 199 + 60 \quad \underline{4} \\ -8 + 36 - 46 + 15 \\ \hline 0 - 32 + 144 - 184 + 60 \end{array}$$

Hence 4 is a root, but 2 is not a root.

As the only possible integral root was removed, the depressed equation

$$8x^3 - 36x^2 + 46x - 15 = 0, \text{ can have no integral roots.}$$

$$\text{Dividing by 8,} \quad x^3 - \frac{9}{2}x^2 + \frac{23}{4}x - \frac{15}{8} = 0.$$

$$\text{Let } x = \frac{y}{2}, \text{ then } y^3 - 9y^2 + 23y - 15 = 0.$$

The possible integral roots are 1, 3, 5.

$$f(1) = 1 - 9 + 23 - 15 = 0.$$

Hence we have $y = 1$, and, by division, $y^2 - 8y + 15 = 0$.

$$\text{Or,} \quad (y - 3)(y - 5) = 0.$$

$$\text{That is,} \quad y = 1, 3, \text{ or } 5.$$

Hence the required four roots are $4, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$.

EXERCISE 166

Find all commensurable, or, if possible, all roots of the following equations:

1. $x^3 - 49x - 120 = 0.$
2. $x^3 - 18x^2 + 87x - 110 = 0.$
3. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$
4. $x^4 - 45x^2 - 40x + 84 = 0.$
5. $8x^3 + 34x^2 - 79x + 30 = 0.$
6. $3x^3 - 8x^2 + 3x + 2 = 0.$
7. $4x^3 - 3x^2 - 25x - 6 = 0.$
8. $2x^3 - 15x^2 + 31x - 12 = 0.$
9. $2x^3 - 13x^2 + 13x + 10 = 0.$
10. $3x^3 + 16x^2 + 23x + 6 = 0.$
11. $x^3 - \frac{14}{3}x^2 + 7x - \frac{10}{3} = 0.$
12. $x^3 - 14x^2 - 5x + 58 = 0.$
13. $x^3 - 13x^2 + 49x - 45 = 0.$
14. $x^4 + x^3 - 24x^2 + 43x - 21 = 0.$
15. $x^4 + 8x^3 - 13x^2 - 92x + 96 = 0.$
16. $24x^3 - 26x^2 + 9x - 1 = 0.$
17. $2x^3 - 11x^2 + 17x - 6 = 0.$
18. $x^4 + 19x^3 + 123x^2 + 305x + 200 = 0.$
19. $2x^4 - 11x^3 + 16x^2 - x - 6 = 0.$
20. $3x^4 - 20x^3 + 41x^2 - 20x - 12 = 0.$
21. $3x^4 - 19x^3 + 39x^2 - 29x + 6 = 0.$
22. $6x^4 - 13x^3 + 20x^2 - 37x + 24 = 0.$
23. $x^4 + 29x^3 + 287x^2 + 1147x + 1560 = 0.$
24. $x^5 - 3x^4 - 8x^3 + 24x^2 - 9x + 27 = 0.$

INCOMMENSURABLE ROOTS

563. Graphic method. The simplest method for determining the incommensurable roots of an equation is the graphic method.

The essential features of this method were discussed in Chapter XVII. The theory of equations, however, sometimes simplifies the work. Thus, we may use synthetic division for finding the various values of a function, we may determine the limits to the roots, we may transform the equation, etc.

Ex. Find graphically the roots of the equation

$$24x^3 - 26x^2 + 9x - 1 = 0.$$

By Descartes' Rule there are no negative roots, but there must be at least one positive real root (§ 555). The superior limit is $1\frac{1}{2}$.

To avoid fractions, we multiply the roots by 10; i.e. make $y = 10x$,

or,
$$x = \frac{y}{10}.$$

$$24y^3 - 260y^2 + 900y - 1000 = 0.$$

Dividing by 4,
$$6y^3 - 65y^2 + 225y - 250 = 0.$$

$$f(0) = -250, f(1) = 6 - 65 + 225 - 250 = -84.$$

By synthetic division we find

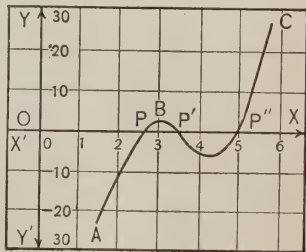
$$f(2) = -12, f(3) = +2, f(4) = -6, f(5) = 0, f(6) = 56.$$

Locating the points

$$(2, -12), (3, 2), (4, -6), (5, 0), (6, 56),$$

and joining, produces the graph ABC , which intersects the x -axis in $P, P',$ and P'' . By measuring OP and OP' , we obtain the approximate roots 2.5 and 3.3, while 5 is an exact root.

Hence the required roots are .25, .33, and .5.



NOTE. To find more exact values of x , the portion of the diagram which contains P_1 or P_2 should be drawn on a larger scale (see page 279).

EXERCISE 167

Find graphically the roots of the following equations:

1. $x^3 + 8x^2 + 19x + 13 = 0$.

2. $x^4 + 4x^3 + 18 = 0$.

3. $x^3 - 13x^2 + 38x + 17 = 0$.

4. $100x^3 + 4x - 1 = 0$.

5. $64x^3 - 16x^2 - 5x + \frac{1}{2} = 0$.

If $f(x) = x^4 - 6x^3 + 7x^2 + 6x - 4$.

6. Find the approximate values of the roots of $f(x) = 0$.

7. Solve $f(x) = 4$.

8. Determine the number of real roots of the equations $f(x) = 1$, $f(x) = 100$, $f(x) = -50$.

9. Solve the system: $\begin{cases} y = f(x), \\ 2x - y = 0. \end{cases}$

10. Solve the system: $\begin{cases} y = f(x), \\ y - 4x = 8. \end{cases}$

If $f(x) = x^5 - x^4 - 11x^3 + 9x^2 + 18x - 4$.

11. Solve $f(x) = 0$.

12. Solve $f(x) = -4$.

13. Determine the number of real and imaginary roots of $f(x) = -20$, $f(x) = 5$, $f(x) = 20$, $f(x) = 36$, $f(x) = 100$.

14. Solve the system: $\begin{cases} y = f(x), \\ y + 5x = 0. \end{cases}$

Solve :

15. $x^x = 10$.

17. $\begin{cases} x^2 + y^2 = 25, \\ x^3 - 3x + y = 1. \end{cases}$

16. $\begin{cases} y = x^3 - 2x + 2, \\ x + y = 4. \end{cases}$

18. $\begin{cases} 4x^2 + y^2 = 4, \\ x^2 + 4x - y = 3. \end{cases}$

19. $\begin{cases} x^2 + y^2 = 8, \\ x + y = 3. \end{cases}$

20. Has the system $\begin{cases} x^2 + y^2 = 8 \\ x + y = 8 \end{cases}$ any real roots?

564. Roots diminished by a given number.

To transform an equation into another one whose roots are less by h than the roots of the given equation.

Let $a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$ (1)

be the given equation.

Make $y = x - h$, then $x = y + h$.

Substituting in (1),

$$a_0(y+h)^n + a_1(y+h)^{n-1} + \dots + a_{n-1}(y+h) + a_n = 0. \quad (2)$$

If we should expand all parentheses, and collect equal powers of y , we should obtain an equation of the form

$$q_0y^n + q_1y^{n-1} + q_2y^{n-2} + \dots + q_n = 0. \quad (3)$$

The work of expanding the parentheses is, however, very tedious. It is simpler to find the coefficients $q_0, q_1, q_2 \dots q_n$ by the following consideration.

Since $y = x - h$, equation (3) may be written

$$q_0(x-h)^n + q_1(x-h)^{n-1} + q_2(x-h)^{n-2} + \dots + q_{n-1}(x-h) + q_n = 0. \quad (4)$$

But this equation is the original equation (1) written in a different form. Therefore any division applied to (4) will give the same quotient and the same remainder as if applied to (1).

But the coefficients q_1, q_2, \dots can be obtained from (4) and hence from (1) by the following operations:

Divide (4) by $x - h$, then

$$\text{Quotient} = q_0(x - h)^{n-1} + q_1(x - h)^{n-2} + \dots + q_{n-1};$$

$$\text{Remainder} = q_n.$$

Dividing the last quotient by $x - h$, we have

$$\text{Quotient} = q_0(x - h)^{n-2} + q_1(x - h)^{n-3} + \dots + q_{n-2};$$

$$\text{Remainder} = q_{n-1}.$$

Evidently, by successive division of the quotients by $x - h$, we should obtain the coefficients $q_n \dots q_1$ as remainders, while

$$q_0 = a_0.$$

565. *To diminish the roots of an equation $f(x) = 0$ by h , divide $f(x)$ by $x - h$ until the remainder does not involve x . This remainder will be the last coefficient (q_n) of the required equation. The remainder obtained by dividing the quotient by $x - h$ is the coefficient q_{n-1} , etc.*

Ex. 1. Find an equation whose roots are less by 2 than the roots of the equation $3x^4 - 20x^3 + 5x^2 + 13x + 70 = 0$.

The division may be arranged as follows:

$$\begin{array}{r}
 3 \quad -20 \quad + 5 \quad + 13 \quad + 70 \quad | \quad 2 \\
 \hline
 \quad + 6 \quad -28 \quad - 46 \quad -66 \\
 \hline
 3 \quad -14 \quad -23 \quad -33 \quad | \quad + 4 \\
 \quad + 6 \quad -16 \quad -78 \\
 \hline
 3 \quad -8 \quad -39 \quad | \quad -111 \\
 \quad + 6 \quad -4 \\
 \hline
 3 \quad -2 \quad | \quad -43 \\
 \quad + 6 \\
 \hline
 3 \quad + 4
 \end{array}$$

Hence the required equation is

$$3x^4 + 4x^3 - 43x^2 - 111x + 4 = 0.$$

EX. 2 Find an equation whose roots are greater by 3 than the roots of the equation $2x^4 - 3x^2 + 5x - 7 = 0$.

Evidently, $h = -3$.

$$\begin{array}{r}
 2 \quad + \quad 0 \quad - \quad 3 \quad + \quad 5 \quad - \quad 7 \quad | \quad -3 \\
 \hline
 \quad - \quad 6 \quad + \quad 18 \quad - \quad 45 \quad + \quad 120 \\
 2 \quad - \quad 6 \quad + \quad 15 \quad - \quad 40 \quad | \quad +113 \\
 \hline
 \quad - \quad 6 \quad + \quad 36 \quad - \quad 153 \\
 2 \quad - \quad 12 \quad + \quad 51 \quad | \quad -193 \\
 \hline
 \quad - \quad 6 \quad + \quad 54 \\
 2 \quad - \quad 18 \quad | \quad +105 \\
 \hline
 \quad - \quad 6 \\
 2 \quad - \quad 24
 \end{array}$$

The required equation is $2x^4 - 24x^3 + 105x^2 - 193x + 113 = 0$.

566. The preceding method can be used to transform an equation into another one in which the second term is wanting.

Let
$$a_0x^n + a_1x^{n-1} + \dots + a_n = 0 \quad (1)$$

be the equation.

Diminishing the roots by h , we have

$$y = x - h, \text{ or } x = y + h.$$

Substituting in (1)

$$a_0(y+h)^n + a_1(y+h)^{n-1} + \dots + a_n = 0.$$

If we should expand and combine equal powers of y , we should obtain $a_0nh + a_1$ as the coefficient of y^{n-1} .

Making this coefficient equal to zero, we obtain

$$h = -\frac{a_1}{na_0}.$$

Hence, if we diminish the roots of (1) by $-\frac{a_1}{na_0}$, the transformed equation is the required one.

567. If the equation is in its simplest form (*i.e.* $a_0 = 1$), diminish the roots by $-\frac{a_1}{n}$; that is, by $-\frac{1}{3}$ the second coeffi-

cient for a cubic, by $-\frac{1}{4}$ the second coefficient for an equation of the fourth degree, etc.

Ex. 3. Transform the equation

$$x^4 - 2x^3 - 6x^2 + 3x + 2 = 0$$

into another one whose second term is wanting.

Diminish the roots by $-\frac{1}{4}$ of -2 , *i.e.* .5.

1	- 2.	- 6.	+ 3.	+ 2.	<u>1.5</u>
	.5	- .75	- 3.375	- .1875	
1	- 1.5	- 6.75	- .375	+ 1.8125	
	.5	- .5	- 3.625		
1	- 1.	- 7.25	- 4.000		
	.5	- .25			
1	- .5	- 7.50			
	.5				
1	0				

Hence the transformed equation is

$$y^4 - 7.5y^2 - 4y + 1.8125 = 0.$$

EXERCISE 168

Transform the following equations into others whose roots are less by h :

- | | |
|---|------------|
| 1. $2x^4 - 4x^3 + 5x^2 - 3x + 1 = 0,$ | $h = 2.$ |
| 2. $3x^4 - 19x^3 + 22x^2 - 17x - 44 = 0,$ | $h = 1.$ |
| 3. $4x^3 - 20x^2 - 17x + 12 = 0,$ | $h = 3.$ |
| 4. $2x^4 - 3x^2 - 9x + 71 = 0,$ | $h = 4.$ |
| 5. $x^5 - 6x^4 + 12x^3 - 12 = 0,$ | $h = 5.$ |
| 6. $x^4 - 12x^3 + 7x^2 - 3x + 12 = 0,$ | $h = 3.$ |
| 7. $x^5 - 1 = 0,$ | $h = 2.$ |
| 8. $x^5 + x^4 + x^2 - 1 = 0,$ | $h = 2.$ |
| 9. $x^3 - 2x^2 + 1 = 0,$ | $h = 1.5.$ |

Transform the following equations into others whose roots are greater by h :

$$10. \quad x^4 + 2x^2 - 17x + 12 = 0, \quad h = 2.$$

$$11. \quad x^5 - 4x^4 + 3x^3 + 2x^2 + 3x + 9 = 0, \quad h = 1.$$

Transform the following equations into others, in which the second term is wanting:

$$12. \quad x^4 - 4x^3 + 2x^2 - 7 = 0. \quad 15. \quad x^3 + 4x^2 - 7x + 2 = 0.$$

$$13. \quad x^3 - 9x^2 + 5x - 3 = 0. \quad 16. \quad x^4 - 8x^3 + 2x - 7 = 0.$$

$$14. \quad x^3 + 6x^2 - 3x - 3 = 0. \quad 17. \quad x^4 - 2x^3 + x^2 - x + 1 = 0.$$

568. Horner's method of approximation. By Horner's method, incommensurable roots may be found to any degree of accuracy.

Let it be required to find the positive root or roots of

$$x^3 - 3x - 4 = 0. \quad (1)$$

First locate the roots either by the graphical method or by § 451.

According to Descartes' Rule, the equation cannot have more than one positive root, and by § 555 it must have one positive root. By substitution we find

$$f(0) = -4, \quad f(1) = -6, \quad f(2) = -2, \quad f(3) = 14.$$

Hence the required root must lie between 2 and 3.

Diminishing the roots of the equation by 2, we obtain

1	+ 0	- 3	- 4	2
	+ 2	+ 4	+ 2	
1	+ 2	+ 1	- 2	
	+ 2	+ 8		
1	+ 4	+ 9		
	+ 2			
1	+ 6			

The transformed equation is

$$y^3 + 6y^2 + 9y - 2 = 0, \quad (2)$$

in which $y = x - 2$.

Since x lies between 2 and 3, y must lie between 0 and 1. Hence y^3 and y^2 are smaller than y , and we may obtain a rough approximation of y by neglecting the first two terms of equation (2).

I.e. $9y = 2$, or $y = .2^+$.

If $y = .2$, we find $f(y) = +.048$; if $y = 0$, $f(y) = -2$.

The value of y must therefore lie between 0 and .2, or $y < .2$.

Substituting $y = .1$, we obtain $f(y) = -1.039$.

Hence y lies between .1 and .2, or $y = .1^+$, *i.e.* $x = 2.1^+$.

Diminishing the roots of (2) by .1, we have

1	+ 6.	+ 9.	- 2.	.1
	.1	.61	.961	
1	6.1	9.61	- 1.039	
	.1	.62		
1	6.2	10.23		
	.1			
1	6.3			

The second transformed equation is

$$z^3 + 6.3z^2 + 10.23z - 1.039 = 0, \quad (3)$$

in which $z = y - .1$, *i.e.* $z < .1$.

Consequently we obtain an approximate value of z by neglecting z^2 and z^3 in equation (3).

$$10.23z = 1.039, \text{ or } z = .1.$$

But as this value is too large, we assume $z = .09$.

Diminishing the roots of (3) by .09,

1	6.3	10.23	-1.039	<u>.09</u>
	.09	.5751	.972459	
1	6.39	10.8051	-	.066541
	.09	.5832		
1	6.48	11.3883		
	.09			
1	6.57			

The third transformed equation is

$$u^3 + 6.57 u^2 + 11.3883 u - .066541 = 0, \quad (4)$$

in which $u = z - .09$, i.e. $u < .01$.

Hence we obtain an approximate value of u from the equation

$$11.3883 u = .066541.$$

$$u = .005+.$$

Diminishing the roots of (4) by .005,

1	6.57	11.3883	-.066541	<u>.005</u>
	.005	.032875	.057105875	
1	6.575	11.421175	-	.009435125
	.005	.032900		
1	6.580	11.454075		
	.005			
1	6.585			

The fourth transformed equation is

$$v^3 + 6.585 v^2 + 11.454075 v - .009435125 = 0, \quad (5)$$

in which $v = u - .005$, or $v < .001$.

Hence v^3 and v^2 are so much smaller than v , that usually three significant places of v may be found by solving the equation

$$11.454075 v = .009435125. \quad (6)$$

Therefore $v = .000823$.

Whence $x = 2.195823$.

569. The work is usually arranged as follows:

1	0.	— 3.	— 4.	<u>2.195823</u>
	2.	4.	2.	
1	2.	1.	— 2.	
	2.	8.		
1	4.	9.		
	2.			
1	6.	9.	— 2.	<u>.1</u>
	.1	.61	.961	
1	6.1	9.61	— 1.039	
	.1	.62		
1	6.2	10.23		
	.1			
1	6.3	10.23	— 1.039	<u>.09</u>
	.09	.5751	.972459	
1	6.39	10.8051	— .066541	
	.09	.5832		
1	6.48	11.3883		
	.09			
1	6.57	11.3883	— .066541	<u>.005</u>
	.005	.032875	.057105875	
1	6.575	11.421175	— .009435125	
	.005	.032900		
1	6.580	11.454075		
	.005			
1	6.585			

$$v = \frac{.009435125}{11.454075} = .000823.$$

570. In finding the approximate value of the *second figure* of the root (*i.e.* y), the student should prove the location of the root by actual substitution. The signs of two functions, including a root, should be opposite (with the rare exception of two nearly equal roots).

571. Any figure after the second can usually be found by dividing the last term with its sign changed, by the coefficient of x . If it is at all doubtful which of two successive numbers should be assumed, take first the greater one, and if this should be too large, the fact can be recognized by the following two rules:

(a) In any transformed equation after the first, the signs of the last two terms must be opposite. If they should be the same, the value assumed for the root is too large.

(b) Unless the equation has two nearly equal roots, the signs of the last terms of all transformed equations must be the same.

572. After the equation has been transformed four or more times, several additional decimals can be found by division of the last terms.

In equation (5) we may express the true value of v as follows:

$$v = \frac{.009435125}{11.454075} - \frac{6.585 v^2 + v^3}{11.454075}.$$

We assumed for v only the first term of the right member, hence the error is equal to

$$\frac{6.585 v^2 + v^3}{11.454075}.$$

A simple calculation shows that this error for $v = .0008$ is less than .000001, an error which cannot affect the first six decimals of the root.

Therefore the three decimals obtained by division are correct.

573. To avoid decimals, multiply the roots of the first transformed equation by 10; *i.e.* multiply the second coefficient by 10, the third by 100; etc.

The roots of the second transformed equation obtained therefrom, are again multiplied by 10, etc.

574. The work may then be arranged as follows:

1	0	— 3	— 4	<u>2.195823</u>
	2	4	2	
1	2	1	— 2	
	2	8		
1	4	9		
	2			
1	60	900	— 2000	<u>1</u>
	1	61	961	
1	61	961	— 1039	
	1	62		
1	62	1023		
	1			
1	630	102300	— 1039000	<u>9</u>
	9	5751	972459	
1	639	108051	— 66541	
	9	5832		
1	648	113883		
	9			
1	6570	11388300	— 66541000	<u>5</u>
	5	32875	57105875	
1	6575	11421175	— 9435125	
	5	32900		
1	6580	11454075		
	5			
1	6585			

$$\frac{9435125}{11454075} = .823.$$

575. If we wish to determine only a certain number of decimals, the work may be further contracted by omitting all figures which do not influence these decimals.

576. Incommensurable negative roots can be found by changing the signs of all roots by the transformation of § 531. and determining the corresponding positive roots.

Ex. By Horner's method, find the real value of $\sqrt[3]{-3}$ to three places of decimals.

Let $x = \sqrt[3]{-3}$.

Then $x^3 + 3 = 0$. Since the required root is negative, change the sign of the roots, *i.e.* $y^3 - 3 = 0$.

1	0	0	-3	1
	1	1	1	
1	1	1	-2	
	1	2		
1	2	3		
	1			
1	30	300		4
	4	130	44	
1	34	430	256	
	4	152		
1	38	588		
	4			
1	520	58800	-256000	4
	4	2096	+243584	
1	524	60896	-12416	
	4	2112		
1	528	63008		

$\sqrt[3]{3} = 1.441+$. Hence $y = 1.441+$ and $x = -1.441+$.

EXERCISE 169

In the following equations, compute to three decimal places the root which lies between the indicated limit:

1. $x^3 - 15x^2 + 63x - 50 = 0$, root between 1 and 2.
2. $x^3 - 12x^2 + 45x - 53 = 0$, root between 2 and 3.
3. $x^3 - 12x^2 + 45x - 53 = 0$, root between 3 and 4.
4. $x^3 - 12x^2 + 57x - 94 = 0$, root between 3 and 4.
5. $x^3 - 12x - 28 = 0$, root between 4 and 5.
6. $x^3 - 15x^2 + 72x - 109 = 0$, root between 6 and 7.
7. $x^3 - 13x^2 + 38x + 17 = 0$, root between 7 and 8.
8. $x^3 - 12x^2 + 35x - 73 = 0$, root between 9 and 10.
9. $x^3 + 2x^2 + 3x - 52 = 0$, root between 2 and 3.

Find to three decimal places the real root of the following equations:

- | | |
|-----------------------------|---------------------------|
| 10. $x^3 - 12x - 132 = 0$. | 13. $x^3 + 7x - 11 = 0$. |
| 11. $x^3 + 2x - 4 = 0$. | 14. $2x^3 + 3x + 4 = 0$. |
| 12. $x^3 + 3x - 2 = 0$. | |

Compute the roots to five decimal places:

- | | |
|---|------------------------------|
| 15. $x^4 - 4x^3 + 18 = 0$, | root between 3 and 4. |
| 16. $x^4 - 4x^3 + 18 = 0$, | root between 2 and 3. |
| 17. $x^4 + 8x^2 + 16x - 440 = 0$, | root between -4 and -5 . |
| 18. $x^4 - 2x^3 - 9x^2 + 10x + 7 = 0$, | root between -2 and -3 . |
| 19. $x^4 - 17 = 0$, | root between 2 and 3. |

Find by Horner's method to three decimal places the value of:

- | | | |
|---------------------|---------------------|-----------------------|
| 20. $\sqrt[3]{3}$. | 21. $\sqrt[4]{7}$. | 22. $\sqrt[3]{-11}$. |
|---------------------|---------------------|-----------------------|

23. The number of cubic feet contained in a cube exceeds the number of feet in all its edges by 17. Find the edge of the cube to three decimal places.

24. Find two numbers differing by 4 whose product is equal to 1386 divided by their sum.

25. The number of cubic feet contained in a cube exceeds the number of square feet in its surface by 100. Find the edge of the cube to three decimal places.

GRAPHIC SOLUTION OF CUBIC EQUATIONS*

577. In the following section, a graphic method for solving cubic equations is given, which is somewhat shorter and more convenient than the usual graphic method. It consists in solving equations by means of straight lines and one standard curve which is identical for all cubics.

The method was applied to quadratics in § 330, and it can also be applied to equations of the fourth degree.†

$$\mathbf{578.} \text{ Consider the cubic } ax^3 + bx + c = 0. \quad (1)$$

$$\text{Let} \quad \quad \quad y = x^3. \quad (2)$$

$$\text{Then} \quad \quad \quad ay + bx + c = 0. \quad (3)$$

The solution of the system (2), (3) for x produces the required roots.

But the graph of (3) is a straight line, while the graph of (2) is identical for all cubic equations. Hence after the graph $y = x^3$ (*AOP* in the diagram) has been constructed, any cubic may be solved by the construction of a straight line.

$$\text{Ex. 1. Solve } 4x^3 - 39x + 35 = 0. \quad (1)$$

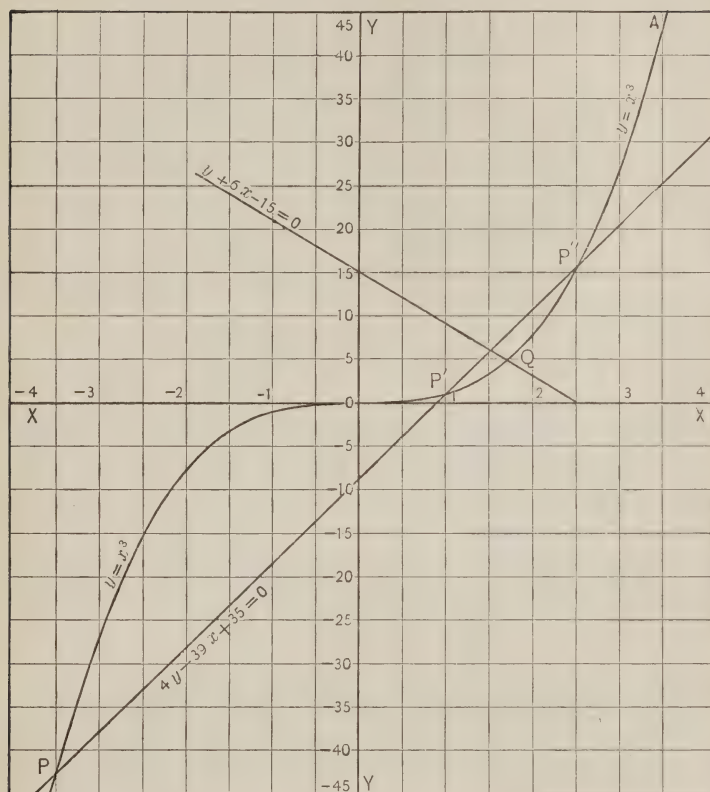
$$\text{Let} \quad \quad \quad y = x^3. \quad (2)$$

$$\text{Then} \quad \quad \quad 4y - 39x + 35 = 0. \quad (3)$$

In (3), if $x = 0$, then $y = -8\frac{3}{4}$, and if $x = 4$, then $y = 30\frac{1}{4}$. The line joining $(0, -8\frac{3}{4})$ and $(4, 30\frac{1}{4})$ intersects the graph of (2) in P , P' , and P'' . By measuring the abscissas of P , P' , and P'' , we find $x = -3\frac{1}{2}$, or $+1$, or $2\frac{1}{2}$.

* The remaining sections of the book are not absolutely necessary for the Examinations of the College Entrance Examination Board.

† The following section is taken from a paper read by the author before the American Mathematical Society in April, 1905.



579. In the equation $ay + bx + c$, if $x = 0$, then $y = -\frac{c}{a}$; if $y = 0$, then $x = -\frac{c}{b}$. Hence, by taking on the x -axis the point $-\frac{c}{b}$, on the y -axis the point $-\frac{c}{a}$, and applying a straight edge, the roots of the equation $ax^3 + bx + c = 0$ can frequently be determined by inspection. If the two points thus constructed on the axis lie very closely together, the drawing is likely to be inaccurate, and it is better to locate one or both points outside the axis.

Ex. 2. Solve $x^3 + 6x - 15 = 0$. (1)

Let $y = x^3$. (2)

Then $y + 6x - 15 = 0$. (3)

Hence the distances cut off by (3) on the x - and y -axes are respectively $2\frac{1}{2}$ and 15, and the line (3) is easily constructed. As there is only one point of intersection, Q , the equation has only one real root, viz. 1.7^+ .

580. The curved line which represents the equation $y = x^3$ is a **cubic parabola**.

The curve $y = x^2$ used in § 330 is a **parabola**.

581. Solution for large roots. Since a linear equation is represented by a straight line, whether the abscissas and ordinates are drawn on the same scale or different scales, we can use the same diagram of the cubic parabola $y = x^3$ for the finding of large and small roots.

For, in the diagram we can assign any values to the abscissas, provided the corresponding ordinates are the cubes of the abscissas.

582. Thus, after the cubic parabola $y = x^3$ has been drawn, we may multiply the numbers on the x -axis by any convenient number, *e.g.* 3, and the values of the ordinates by the cubes of the number, *i.e.* 27.

NOTE. Similarly in the diagram on page 309, the value of the abscissas may be multiplied by any convenient number, *e.g.* 10, provided the ordinates are multiplied by the square of that number, *i.e.* 100. The diagram may then be used for roots between -60 and $+60$.

Ex. 3. Solve graphically $x^3 + 2x - 320 = 0$. (1)

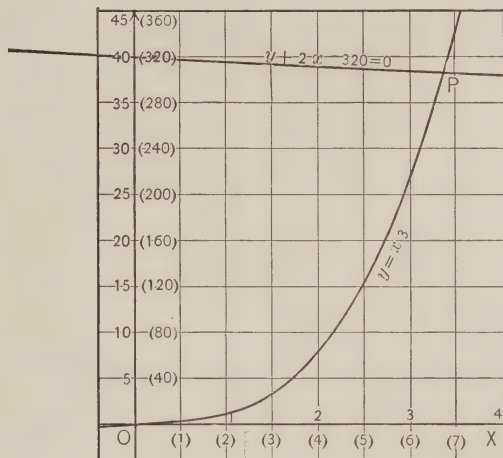
The superior limit is $\sqrt[3]{320}$ or 7^+ , and there can be no negative roots. Hence multiply the values of the abscissas in the diagram by 2; then the values of the ordinates have to be multiplied by 8. (The resulting values are given in parentheses.)

Let $y = x^3$. (2)

Then $y + 2x - 320 = 0$, or $y = 320 - 2x$. (3)

If $x = 0$, $y = 320$, and if $x = 8$, $y = 304$.

Joining the points $(0, 320)$ and $(8, 304)$, we obtain the real root 6.8-, while the other roots are imaginary.



NOTE. The student should draw the graph of $y = x^3$ from $x = -3\frac{1}{2}$ to $x = +3\frac{1}{2}$ (or from -4 to $+4$) on a large scale, and use one curve for the solution of a number of equations. The table on page 288 will be found useful for the construction, as it gives the cubes of integers and fractions; thus 2.3^3 is found to be 12.167.

EXERCISE 170

Find graphically the real roots of the following equations:

- | | |
|---------------------------|----------------------------|
| 1. $x^3 + 4x - 16 = 0$. | 5. $x^3 - 7x + 6 = 0$. |
| 2. $x^3 - 5x - 12 = 0$. | 6. $4x^3 - 39x - 35 = 0$. |
| 3. $x^3 - 2x + 4 = 0$. | 7. $x^3 - 5x + 20 = 0$. |
| 4. $2x^3 - 9x + 27 = 0$. | 8. $x^3 - 5x - 15 = 0$. |

- | | |
|---------------------------|------------------------------|
| 9. $x^3 - 5x - 5 = 0.$ | 17. $x^3 + 10x - 13 = 0.$ |
| 10. $x^3 - 32x - 80 = 0.$ | 18. $x^3 - 45x - 152 = 0.$ |
| 11. $2x^3 - 5x + 20 = 0.$ | 19. $x^3 - 60x + 180 = 0.$ |
| 12. $x^3 + 8x - 64 = 0.$ | 20. $x^3 - 90x + 340 = 0.$ |
| 13. $x^3 - 10x - 48 = 0.$ | 21. $x^3 - 75x - 250 = 0.$ |
| 14. $x^3 - 9x + 54 = 0.$ | 22. $x^3 - 100x + 500 = 0.$ |
| 15. $x^3 - 14x + 24 = 0.$ | 23. $x^3 + 120x - 560 = 0.$ |
| 16. $x^3 - 30x - 18 = 0.$ | 24. $x^3 - 200x + 1200 = 0.$ |

Transform into equations without x^2 , and solve graphically:

$$25. \quad x^3 - 6x^2 + 11x - 6 = 0.$$

$$26. \quad x^3 + 3x^2 - 4x + 1 = 0.$$

$$27. \quad x^3 + 9x^2 - 7 = 0.$$

Determine graphically the character of the roots of the following equations:

$$28. \quad x^3 - 3x - 5 = 0.$$

$$30. \quad x^3 - 3x + 5 = 0.$$

$$29. \quad x^3 + 3x - 5 = 0.$$

$$31. \quad x^3 + 3x + 5 = 0.$$

583. Graphic representation of a cubic function.

Consider the equation

$$x^3 + px + q = 0. \tag{1}$$

$$\text{Let} \quad y = x^3. \tag{2}$$

$$\text{Then} \quad y + px + q = 0, \tag{3}$$

$$\text{or} \quad y = -px - q.$$

In the annexed diagram, let COD represent the cubic parabola $y = x^3$, and BE the straight line $y + px + q = 0$, or $y = -px - q$.

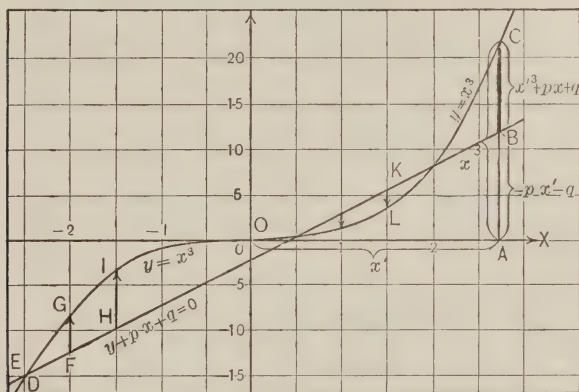
Let OA or x' be any particular value of x ,

then $CA = x'^3$,

and $BA = -px' - q$.

Hence $CB = CA - BA = x'^3 + px' + q$.

I.e. the value of the function $x^3 + px + q$ for any particular value x' is represented by that part of the corresponding ordinate which is intercepted by the straight line $y + px + q = 0$, and the



cubic parabola $y = x^3$. The distance is measured from the straight line, and is taken positive if it extends upward, negative if it extends downward.

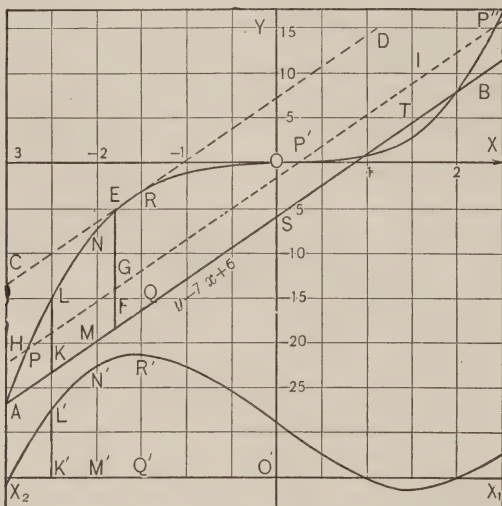
Thus in the annexed diagram, $f(x) = x^3 - \frac{21}{4}x + \frac{5}{2}$, and we have $f(-2) = FG = 5$, $f(-1\frac{1}{2}) = HI = 7$, $f(1\frac{1}{2}) = KL = -2$, etc.

Ex. 1. Find the greatest value of the function $x^3 - 7x + 6$, for a negative x . (See diagram on page 490.)

Construct AB , the locus of $y - 7x + 6 = 0$. Draw CD parallel to AB , touching the cubic parabola in E , then FE , or 14, is the required value.

Ex. 2. Which values of x will make the function $x^3 - 7x + 6$ equal to 4, i.e. $x^3 - 7x + 6 = 4$?

On any ordinate, from the straight line AB , lay off 4 units upwards, as FG . Through G draw HI parallel to AB , intersecting the cubic parabola in $P, P',$ and P'' . By measuring the abscissas of $P, P',$ and P'' , we find $x = -2\frac{3}{4},$ or $\frac{1}{4},$ or $2\frac{1}{2}$.



NOTE. If we consider the distances cut off from SB by the ordinates, as abscissas, *e.g.* $ST = 1\frac{1}{2}$, then the cubic parabola represents the function $x^3 + px + q$ in so-called "oblique coördinates."

584. To construct the graph of $x^3 + px + q$ in the usual manner ("rectangular coördinates"), make $K'L' = KL$, $M'N' = MN$, $O'R' = OR$, etc. The curve $L'N'R'$ is the required graph of $x^3 + px + q$.

585. The value of the function $ax^3 + bx + c$ is equal to a times the part of the corresponding ordinate which is intercepted by the straight line $ay + bx + c = 0$, and the cubic parabola $y = x^3$.

The proof is similar to that of § 585.

NOTE. This method of constructing graphically the value of a function may be applied in a similar manner to quadratics.

EXERCISE 171

Find graphically :

1. The value of $x^3 + 4x - 16$, if x equals $-3, -2.5, -2.1, 3.5$.
2. The value of $x^3 + 4x - 8$, if x equals $-1.6, -1.5, 2, 1.5$.
3. The value of $x^3 - 6x - 15$, if $x = -3, -2, 1.5, 3.5$.
4. The value of $x^3 - 5x + 18$, if $x = -8, -5, +3, +7$.
5. The value of x , if $x^3 - 5x - 12 = 5$.
6. The value of x , if $x^3 - 5x - 12 = -10$.
7. The value of x , if $x^3 - 5x - 12 = -40$.
8. The value of x , if $x^3 - 5x - 12 = 10$.
9. The smallest value of $x^3 - 5x - 12$ for a positive x .
10. The greatest value of $x^3 - 5x + 10$ for a negative x .
11. Construct the graph of $x^3 - 12x - 30 = 0$.
12. Construct the graph of $x^3 - 8 = 0$.

Find :

13. The value of $2x^3 + 9x + 20 = 0$, if x equals $3, 2.5, -1.5$.
14. The value of $3x^3 + 9x - 25 = 0$, if x equals $-3, -5, -2$.
15. The smallest value of $3x^3 - 9x - 25$, for a positive x .
16. The character of the roots of the equation $x^3 - 7x - 5 = 0$.
17. Locate the roots of the equation $4x^3 - 80x + 570 = 0$.
18. For which values of x is $4x^3 - 80x + 570 = 0$ positive?
19. Construct the locus of $x^3 - 14x + 20 = 0$.
20. Construct the locus of $x^3 - 15x - 20 = 0$.

GENERAL SOLUTION OF CUBIC EQUATIONS

586. The methods used in the preceding chapters relate to numerical equations only, but they cannot be used to find *general solutions*, i.e. the values of the roots of literal equations in terms of the coefficients.

The only general solution given thus far is the solution of the quadratic $ax^2 + bx + c = 0$; (1)

viz.
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \quad (2)$$

This solution is general because any quadratic may be considered a special case of (1), and the roots of any equation may be found by substituting in (2) the particular values of the coefficients in place of a , b , and c .

587. It is possible to give general solutions of equations of the third and fourth degree, but it was first proved by Abel* that it is impossible to obtain general algebraic solutions of equations higher than the fourth degree.

588. The cube roots of unity. To find the cube roots of unity, we have to solve the equation

$$x^3 = 1, \text{ or } x^3 - 1 = 0. \quad (1)$$

Factoring, $(x - 1)(x^2 + x + 1) = 0.$

Hence $x - 1 = 0,$

or $x^2 + x + 1 = 0. \quad (2)$

Whence $x = 1, \text{ or } x = -\frac{1}{2}(1 \pm i\sqrt{3}).$

Let $-\frac{1}{2}(1 + i\sqrt{3}) = \omega. \dagger$

Then $\omega^2 = \frac{1}{4}(1 + i\sqrt{3})^2 = -\frac{1}{2}(1 - i\sqrt{3}).$

Hence the three cube roots of unity are 1, ω , ω^2 .

* For an elementary representation of this proof, see Peterson, *Theorie der Algebraischen Gleichungen*, p. 113. \dagger The Greek letter Omega.

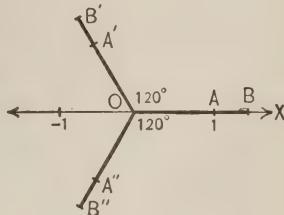
589. If one cube root of any quantity is r , the two others are $r\omega$ and $r\omega^2$.

$$\text{For} \quad \sqrt[3]{a} = \sqrt[3]{a \cdot 1} = \sqrt[3]{a} \cdot \sqrt[3]{1} = r\sqrt[3]{1},$$

i.e. $r, r\omega, r\omega^2$, are the three cube roots of a .

590. In the annexed diagram, the lines OA, OA', OA'' represent the three cube roots of unity graphically in the plane of complex numbers. The length of each line is 1, and the angles included by them are 120° .

If OB represents the real value of $\sqrt[3]{a}$, then OB' and OB'' represent the other two cube roots of a .



591. Cardan's solution of the cubic equation. By § 565, any cubic can be reduced to the form

$$x^3 + px + q = 0. \quad (1)$$

$$\text{Suppose} \quad y = m + n,$$

$$\text{then} \quad y^3 = m^3 + 3mn(m + n) + n^3.$$

Substituting for $m + n$ its value, and transposing,

$$y^3 - 3mny - (m^3 + n^3) = 0. \quad (2)$$

If we make the coefficients of equation (2) equal to the coefficients of (1), the two equations become identical, and $y = x$.

$$\text{Or, if} \quad 3mn = -p, \quad (3)$$

$$\text{and} \quad m^3 + n^3 = -q, \quad (4)$$

$$\text{then} \quad x = m + n. \quad (5)$$

But m and n are easily found from (3) and (4).

$$\text{From (3)} \quad 4m^3n^3 = -\frac{4p^3}{27}. \quad (6)$$

Squaring (4) and subtracting (6),

$$(m^3 - n^3)^2 = q^2 + \frac{4p^3}{27}. \quad (7)$$

Hence
$$m^3 - n^3 = \sqrt{q^2 + \frac{4p^3}{27}}. \quad (8)$$

But
$$m^3 + n^3 = -q.$$

Therefore
$$m^3 = -\frac{q}{2} + \frac{1}{2}\sqrt{q^2 + \frac{4p^3}{27}},$$

or
$$m = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Similarly
$$n = \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

Therefore
$$x = m + n = \sqrt[3]{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} + \sqrt[3]{-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}.$$

This result is known as *Cardan's formula*, although Cardan did not discover it. He published it in 1545 as his own work, after having obtained it under the pledge of secrecy from Tartaglia.

592. Since m and n are cube roots, they have three values, and if r and r' represent respectively one root, then

$$m = r, \text{ or } r\omega, \text{ or } r\omega^2,$$

$$n = r' \text{ or } r'\omega, \text{ or } r'\omega^2.$$

If every value of m could be combined with every value of n , there would be nine values of x ; but since $3mn = -p$, the product of m and the corresponding value of n must be real. Hence each value of m can be combined with only one value of n , thus producing the roots

$$x = r + r',$$

or
$$x = r\omega + r'\omega^2,$$

or
$$x = r\omega^2 + r'\omega.$$

593. If all three roots are real and unequal, Cardan's formula gives the answers as the sum of two cube roots of complex numbers. As there is no general algebraic method for reducing this answer to a real form, this case is known as the *irreducible case*. Horner's method should then be employed.

Ex. 1. Solve the equation

$$x^3 - 6x - 9 = 0. \quad (1)$$

The shortest way is to substitute the values of p and q in Cardan's formula. Since this formula, however, is difficult to remember, we proceed as follows:

$$\text{Let} \quad y^3 = (m + n)^3 = m^3 + 3mn(m + n) + n^3.$$

$$\text{Or} \quad y^3 - 3mny - (m^3 + n^3) = 0. \quad (2)$$

Equating the coefficients of (1) and (2),

$$3mn = 6, \quad (3)$$

$$m^3 + n^3 = 9. \quad (4)$$

$$\text{From (3)} \quad 4m^3n^3 = 32. \quad (5)$$

$$\text{Hence} \quad m^6 - 2m^3n^3 + n^6 = 49. \quad (6)$$

$$\text{Or} \quad m^3 - n^3 = 7.* \quad (7)$$

$$m^3 = 8, \quad n^3 = 1.$$

$$\text{Whence} \quad m = 2, \quad 2\omega, \quad 2\omega^2.$$

$$n = 1, \quad \omega, \quad \omega^2.$$

$$\text{Therefore} \quad x = 3, \quad 2\omega + \omega^2, \quad 2\omega^2 + \omega. \quad (8)$$

$$\text{Substituting the value of } \omega, \quad x = 3, \text{ or } -\frac{1}{2}(3 \pm i\sqrt{3}). \quad (9)$$

* The value -7 will produce the same value of x as $+7$.

EXERCISE 172

Solve by Cardan's method:

- | | |
|--------------------------|-----------------------------------|
| 1. $x^3 - 3x - 2 = 0.$ | 8. $x^3 - 72x - 280 = 0.$ |
| 2. $x^3 - 9x + 28 = 0.$ | 9. $x^3 - 6x + 9 = 0.$ |
| 3. $x^3 - 18x - 35 = 0.$ | 10. $x^3 + 9x - 26 = 0.$ |
| 4. $x^3 - 36x - 91 = 0.$ | 11. $x^3 - 9x - 28 = 0.$ |
| 5. $x^3 - 3x + 2 = 0.$ | 12. $x^3 - 72x + 280 = 0.$ |
| 6. $x^3 - 27x - 54 = 0.$ | 13. $x^3 - 6x^2 - 12x + 112 = 0.$ |
| 7. $x^3 + 9x + 26 = 0.$ | 14. $x^3 + 5x^2 + 8x + 6 = 0.$ |

SOLUTION OF BIQUADRATICS

594. A biquadratic is an equation of the fourth degree.

595. Descartes' solution of the biquadratic.

$$\text{Let } x^4 + px^2 + qx + r = 0. \quad (1)$$

$$\begin{aligned} \text{Assume } x^4 + px^2 + qx + r &= (x^2 + lx + m)(x^2 - lx + n) \\ &= x^4 + (-l^2 + m + n)x^2 - l(m - n)x + mn. \end{aligned} \quad (2)$$

Equating the coefficients, we have

$$-l^2 + m + n = p, \quad -l(m - n) = q, \quad mn = r. \quad (3)$$

Considering that $(m + n)^2 - (m - n)^2 - 4mn = 0$, it is easy to eliminate m and n . We obtain

$$l^6 + 2pl^4 + (p^2 - 4r)l^2 - q^2 = 0. \quad (4)$$

Equation (4) is a cubic in l^2 , which has always one real positive root. When l is known, the values of m and n are determined from equation (3). The solutions of the two equations

$$x^2 + lx + m = 0$$

and

$$x^2 - lx + n = 0$$

produce the required roots.

Ex. Solve $x^4 - 6x^2 + 8x - 3 = 0$.

As $p = -6$, $q = 8$, $r = -3$, l is found from the cubic

$$l^3 - 12l^2 + 48l - 64 = 0.$$

A real root of this equation is $l^2 = 4$, hence $l = \pm 2$. Taking $l = 2$, we find from equation (3) $m = -3$, $n = 1$, and the two quadratic equations are :

$$x^2 + 2x - 3 = 0$$

and

$$x^2 - 2x + 1 = 0.$$

Hence the required roots are 1, 1, 1, and -3 .

EXERCISE 173

Solve by Descartes' method :

1. $x^4 - x^2 + 4x - 4 = 0$.

5. $x^4 - 5x^2 - 10x - 6 = 0$.

2. $x^4 - 4x^2 + 4x - 1 = 0$.

6. $x^4 - 4x^2 + 12x - 9 = 0$.

3. $x^3 - 6x^2 + 3x + 2 = 0$.

7. $x^4 - 25x^2 + 60x - 36 = 0$.

4. $x^4 - 7x^2 - 12x + 18 = 0$.

8. $x^4 - 60x^2 + 40x + 396 = 0$.

RECIPROCAL EQUATIONS

596. An equation is called a **reciprocal equation** if the reciprocal of any root is again a root.

In a reciprocal equation of odd degree, one root must be its own reciprocal, hence one root must be either $+1$ or -1 .

597. Let $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$ (1)

be a reciprocal equation.

Substituting $\frac{1}{x}$ for x ,

$$a_0\left(\frac{1}{x}\right)^n + a_1\left(\frac{1}{x}\right)^{n-1} + \dots + a_{n-1}\left(\frac{1}{x}\right) + a_n = 0, \quad (2)$$

or

$$a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0 = 0. \quad (3)$$

Equation (3) has the same roots as equation (1). Hence the corresponding coefficients are equal or proportional.

$$\text{I.e.} \quad \frac{a_0}{a_n} = \frac{a_n}{a_0}, \text{ or } a_0^2 = a_n^2.$$

Therefore $a_n = \pm a_0$.

Similarly it follows that $a_{n-1} = \pm a_1$, etc.

598. *Either the coefficients of terms equally distant from the first and last terms are equal or their absolute values are equal and their signs opposite.*

$$\text{Thus} \quad 2x^4 - 5x^3 + 2x^2 - 5x + 2 = 0,$$

$$3x^5 - 4x^4 + 2x^3 + 2x^2 - 4x + 3 = 0,$$

$$x^5 + x^4 - x - 1 = 0$$

are reciprocal equations.

599. A reciprocal equation of even degree having the signs of the first and last terms equal is called a **standard reciprocal equation**.

$$3x^4 - 5x^3 + 7x^2 - 5x + 3 = 0 \text{ is a standard reciprocal equation.}$$

600. *Any reciprocal equation can be reduced to a standard reciprocal equation by removing the factors $x-1$ or $x+1$, or both.*

Any reciprocal equation of odd degree has the roots $+1$ or -1 (§ 599), hence by removing the factor $(x-1)$ or $(x+1)$ it can be reduced to even degree.

A reciprocal equation of even degree, with the signs of the first and last terms opposite, can have no middle term. By grouping terms it can easily be shown that $x^2 - 1$ is a factor of the left member. By removing the factor $x^2 - 1$, the equation is reduced to a standard equation.

601. *A standard reciprocal equation can be reduced to an equation of half its dimension.*

$$\text{Let} \quad 2x^6 - 3x^5 + 5x^4 - 7x^3 + 5x^2 - 3x + 2 = 0. \quad (1)$$

Dividing by x^3 ,

$$2x^3 - 3x^2 + 5x - 7 + 5\left(\frac{1}{x}\right) - 3\left(\frac{1}{x^2}\right) + 2\left(\frac{1}{x^3}\right) = 0.$$

Grouping terms,

$$2\left(x^3 + \frac{1}{x}\right) - 3\left(x^2 + \frac{1}{x^2}\right) + 5\left(x + \frac{1}{x}\right) - 7 = 0. \quad (2)$$

Let $y = x + \frac{1}{x}$, then $y^2 = x^2 + 2 + \frac{1}{x^2}$.

Hence $x^2 + \frac{1}{x^2} = y^2 - 2$.

$$\begin{aligned} x^3 + \frac{1}{x^3} &= \left(x + \frac{1}{x}\right)\left(x^2 - 1 + \frac{1}{x^2}\right) \\ &= y(y^2 - 3) = y^3 - 3y. \end{aligned}$$

Substituting in (2)

$$2(y^3 - 3y) - 3(y^2 - 2) + 5y - 7 = 0.$$

Simplifying, $2y^3 - 3y^2 - y - 1 = 0$.

Ex. 1. Solve $6x^5 - 29x^4 + 27x^3 + 27x^2 - 29x + 6 = 0$.

Evidently $x = -1$ is a root.

Dividing by $x + 1$,

$$6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0.$$

Dividing by x^2 and grouping,

$$6\left(x^2 + \frac{1}{x^2}\right) - 35\left(x + \frac{1}{x}\right) + 62 = 0.$$

Let $x + \frac{1}{x} = y$, then $x^2 + \frac{1}{x^2} = y^2 - 2$.

Substituting, $6y^2 - 12 - 35y + 62 = 0$.

Or

$$6y^2 - 35y + 50 = 0.$$

$$y = \frac{35 \pm \sqrt{1225 - 1200}}{12} = 3\frac{1}{3} \text{ or } 2\frac{1}{2}.$$

Hence

$$x + \frac{1}{x} = 3\frac{1}{3}, \text{ or } x + \frac{1}{x} = 2\frac{1}{2}.$$

Solving, we find the roots 1, 3, $\frac{1}{3}$, 2, $\frac{1}{2}$.

EXERCISE 174

Solve:

1. $8x^4 - 54x^3 + 101x^2 - 54x + 8 = 0.$

2. $10x^4 - 77x^3 + 150x^2 - 77x + 10 = 0.$

3. $5x^3 - 2x^2 + 2x - 5 = 0.$

4. $x^4 + 5x^3 - 5x - 1 = 0.$

5. $x^3 - 6x^2 + 6x - 1 = 0.$

6. $2x^4 - 9x^3 + 14x^2 - 9x + 2 = 0.$

7. $4x^4 - 25x^3 + 42x^2 - 25x + 4 = 0.$

8. $6x^4 - 35x^3 + 62x^2 - 35x + 6 = 0.$

9. $12x^4 - 91x^3 + 194x^2 - 91x + 12 = 0.$

10. $5x^4 - 36x^3 + 62x^2 - 36x + 5 = 0.$

11. Find five values of $\sqrt[5]{1}$.

12. Find five values of $\sqrt[5]{-1}$.

APPENDIX

I. MULTIPLICATION BY DETACHED COEFFICIENTS

1. When the literal part of the product of two polynomials can be obtained by inspection, the work of multiplication may be simplified by omitting all literal factors.

Ex. 1. Multiply $4x^2 + 6x + 2$ by $2x^2 - 5x - 1$.

ORDINARY SOLUTION

$$\begin{array}{r}
 4x^2 + 6x + 2 \\
 2x^2 - 5x - 1 \\
 \hline
 8x^4 + 12x^3 + 4x^2 \\
 \quad - 20x^3 - 30x^2 - 10x \\
 \qquad \quad - 4x^2 - 6x - 2 \\
 \hline
 8x^4 - 8x^3 - 30x^2 - 16x - 2.
 \end{array}$$

DETACHED COEFFICIENTS

$$\begin{array}{rrrr}
 4 & + & 6 & + & 2 \\
 2 & - & 5 & - & 1 \\
 \hline
 8 & + & 12 & + & 4 \\
 & - & 20 & - & 30 & - & 10 \\
 & & & - & 4 & - & 6 & - & 2 \\
 \hline
 8 & - & 8 & - & 30 & - & 16 & - & 2.
 \end{array}$$

Hence the product equals:

$$8x^4 - 8x^3 - 30x^2 - 16x - 2.$$

2. An example containing an irregular sequence of exponents may be made regular by the insertion of zero coefficients, *e.g.* $x^3 + 2x + 1 = x^3 + 0x^2 + 2x + 1$.

Ex. 2. Multiply $x^3 + x^2 - 7$ by $2x^2 + 1$.

$$\begin{array}{r}
 1 \quad + 1 \quad + 0 \quad - 7 \\
 2 \quad + 0 \quad + 1 \\
 \hline
 2 \quad + 2 \quad + 0 \quad + 14 \\
 \hline
 \quad \quad + 1 \quad + 1 \quad + 0 \quad - 7 \\
 \hline
 2 \quad + 2 \quad + 1 \quad - 13 \quad + 0 \quad - 7.
 \end{array}$$

The highest exponent of the product is obviously 5, hence the product equals $2x^5 + 2x^4 + x^3 - 13x^2 - 7$.

EXAMPLES

Perform the following multiplications by detached coefficients:

1. $(x^2 + 2x + 5)(x^2 - 3x + 1)$.
2. $(x^2 - 3x - 7)(x^2 + 3x - 2)$.
3. $(2x^3 - 5x^2 + 2x + 1)(x^2 + 3x + 1)$.
4. $(a^3 + 3a^2b + 5ab^2 - b^3)(a^2 - 2ab + b^2)$.
5. $(x^3 + 2x - 5)(x^2 - 5x + 1)$.
6. $(6a^2 - 2ab + 5b^2)(6a^2 + 2ab - 5b^2)$.
7. $(m^3n^3 + 6m^2n^2 + 9)(m^2n^2 + 5mn + 1)$.

3. The method of detached coefficients can be applied to addition, division, extracting of square roots, etc.

II. ADDITIONAL CASES IN FACTORING

A. Trinomials of the type $p^2x^4 + qx^2y^2 + r^2y^4$.

The trinomial $p^2x^4 + qx^2y^2 + r^2y^4$ can be factored if $\pm 2pr - q$ is a perfect square.

Ex. 1. Factor $9x^4 + 15x^2y^2 + 16y^4$.

$$\begin{aligned}
 9x^4 + 15x^2y^2 + 16y^4 &= 9x^4 + 24x^2y^2 + 16y^4 - 9x^2y^2 \\
 &= (3x^2 + 4y^2)^2 - (3xy)^2 \\
 &= (3x^2 + 3xy + 4y^2)(3x^2 - 3xy + 4y^2).
 \end{aligned}$$

Ex. 2. Factor $16x^4 - 56x^2y^2 + 25y^4$.

$$\begin{aligned} 16x^4 - 56x^2y^2 + 25y^4 &= 16x^4 - 40x^2y^2 + 25y^4 - 16x^2y^2 \\ &= (4x^2 - 5y^2)^2 - (4xy)^2 \\ &= (4x^2 + 4xy - 5y^2)(4x^2 - 4xy - 5y^2). \end{aligned}$$

EXAMPLES

Factor:

- | | |
|--------------------------------|--------------------------------|
| 1. $a^4 + a^2 + 1$. | 6. $x^4 + y^4 - 14x^2y^2$. |
| 2. $a^4 + 14a^2 + 81$. | 7. $9x^4 + 11x^2y^2 + 4y^4$. |
| 3. $64a^4 + 12a^2 + 1$. | 8. $9a^4 + 3a^2b^2 + 4b^4$. |
| 4. $9x^4 - 4x^2y^2 + 4y^4$. | 9. $a^4 - 18a^2b^2 + b^4$. |
| 5. $16x^4 - 25x^2y^2 + 9y^4$. | 10. $4m^4 - 28m^2n^2 + 9n^4$. |

B. Binomials of the type $a^n \pm b^n$.*

1. *Binomials of the form $a^n - b^n$ can always be factored.*

By actual division:

$$\frac{a^5 - b^5}{a - b} = a^4 + a^3b + a^2b^2 + ab^3 + b^4.$$

Hence $a^5 - b^5 = (a - b)(a^4 + a^3b + a^2b^2 + ab^3 + b^4)$.

Similarly, $a^6 - b^6 = (a - b)(a^5 + a^4b + a^3b^2 + a^2b^3 + ab^4 + b^5)$.

Or in general,

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + b^{n-1}).$$

2. To obtain prime factors, it is better to consider $a^6 - b^6$ the difference of two squares than the difference of two 6th powers.

$$\begin{aligned} \text{Thus, } a^6 - b^6 &= (a^3 + b^3)(a^3 - b^3) \\ &= (a + b)(a^2 - ab + b^2)(a - b)(a^2 + ab + b^2). \end{aligned}$$

Similarly, $a^{10} - b^{10}$ should be considered the difference of two 5th powers.

$$\begin{aligned} a^{10} - b^{10} &= (a^2 - b^2)(a^8 + a^6b^2 + a^4b^4 + a^2b^6 + b^8) \\ &= (a + b)(a - b)(a^8 \dots). \end{aligned}$$

* See Chapter XVI.

3. Binomials of the form $a^n + b^n$ can be factored if n is odd.

By actual division:

$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

Hence $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.

Similarly,

$$a^7 + b^7 = (a + b)(a^6 - a^5b + a^4b^2 - a^3b^3 + a^2b^4 - ab^5 + b^6).$$

Or in general, if n is odd,

$$a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \dots + b^{n-1}).$$

4. The sum of two even powers, as $a^4 + b^4$, $a^6 + b^6$, etc., is not exactly divisible by $a + b$ or $a - b$.

Frequently such expressions can be considered the sum of two odd powers.

$$a^6 + b^6 = (a^2)^3 + (b^2)^3 = (a^2 + b^2)(a^4 - a^2b^2 + b^4).$$

$$a^{10} + b^{10} = (a^2)^5 + (b^2)^5 = (a^2 + b^2)(a^8 - a^6b^2 + a^4b^4 - a^2b^6 + b^8).$$

EXERCISE

Divide:

1. $x^5 - y^5$ by $x - y$.

5. $a^{12} + b^{12}$ by $a^4 + b^4$.

2. $x^5 + 1$ by $x + 1$.

6. $1 + m^6$ by $1 + m^2$.

3. $m^7 - n^7$ by $m - n$.

7. $a^6 - b^6$ by $a^2 - b^2$.

4. $a^4 - 16$ by $a - 2$.

8. $a^6 - b^6$ by $a - b$.

Factor:

9. $m^5 - n^5$.

15. $32 + m^5n^5$.

21. $a^6 + 1$.

10. $a^7 - b^7$.

16. $m^{10} - 1$.

22. $a^6 + 64$.

11. $c^5 + 32$.

17. $a^6b^6 - c^6$.

23. $a^{10} + 1$.

12. $a^7 - 1$.

18. $a^{12} - b^{12}$.

24. $a^{10} + b^{10}c^{10}$.

13. $a^9 + 1$.

19. $a^9 - 729$.

25. $x^8 - y^8$.

14. $a^5b^5 + c^5$.

20. $a^{12} - a^6$.

III. HIGHEST COMMON FACTOR AND LOWEST COMMON MULTIPLE

1. **The Highest Common Factor of two expressions which cannot be factored by inspection** may be found by a method analogous to the one used in arithmetic for the finding of the greatest common divisor of two numbers. (Euclidean Method.)

Suppose it is required to find the H. C. F. of $x^4 - 5x^3 + 4x^2 + 10x - 12$ and $x^3 - 3x^2 - 3x + 9$.

By dividing the first expression by the second, we obtain the quotient $x - 2$ and the remainder $x^2 - 5x + 6$. Hence

$$(x^4 - 5x^3 + 4x^2 + 10x - 12) = (x^3 - 3x^2 - 3x + 9)(x - 2) + (x^2 - 5x + 6) \quad (1)$$

From (1) follows that any factor contained in the divisor ($x^3 - 3x^2 - 3x + 9$) and the remainder ($x^2 - 5x + 6$) is a factor of the entire right member, and therefore a factor of the left member. Hence a factor common to divisor and remainder is also a common factor of divisor and dividend.

Similarly it follows that a factor common to dividend and divisor is also a common factor of divisor and remainder.

Hence the required H. C. F. of dividend and divisor is also the H. C. F. of divisor and remainder.

We have therefore to find the H. C. F. of

$$x^3 - 3x^2 - 3x + 9 \text{ and } x^2 - 5x + 6.$$

This can be done by determining whether or not the factors of $x^2 - 5x + 6$ are also factors of $x^3 - 3x^2 - 3x + 9$.

We may, however, divide again.

$$\begin{array}{r|l} x^3 - 3x^2 - 3x + 9 & x^2 - 5x + 6 \\ x^3 - 5x^2 + 6x & x + 2 \\ \hline 2x^2 - 9x + 9 & \\ 2x^2 - 10x + 12 & \\ \hline \text{Rem.} = x - 3 & \end{array}$$

It follows in precisely the same manner as before that the required H. C. F. must be the H. C. F. of the new divisor $x^2 - 5x + 6$ and the new remainder $x - 3$.

Dividing again, we obtain no remainder, and therefore $x - 3$ is H. C. F. of $x^2 - 5x + 6$ and $x - 3$.

Hence the required H. C. F. = $x - 3$.

The work of the last example may be arranged as follows :

		Quotients
$x^4 - 5x^3 + 4x^2 + 10x - 12$	$x^3 - 3x^2 - 3x + 9$	$x - 2$
$x^4 - 3x^3 - 3x^2 + 9x$	$x^3 - 5x^2 + 6x$	
$-2x^3 + 7x^2 + x - 12$	$2x^2 - 9x + 9$	$x + 2$
$-2x^3 + 6x^2 + 6x - 18$	$2x^2 - 10x + 12$	
First remainder,		
$x^2 - 5x + 6$	$x - 3$	$x - 2$
$x^2 - 3x$		
$-2x + 6$		
$-2x + 6$		

H. C. F. = $x - 3$.

2. Before dividing, all monomial factors of the two given expressions should be removed and their H. C. F. be determined separately.

All monomial factors of any of the successive divisors should be removed, as they do not affect the answer.

If the first term of a dividend is not exactly divisible by the first term of the divisor, multiply the dividend by such a number as will make the term divisible. *E.g.* the first term of $3x^2 + 5x + 2$ is not exactly divisible by $2x + 3$, hence multiply $3x^2 + 5x + 2$ by 2, and divide $6x^2 + 10x + 4$ by $2x + 3$.

Since the divisors do not contain any monomial factors, the introduction of such a factor into the dividend does not produce a common factor, and hence does not affect the result.

Ex. 2. Find the H. C. F. of $3x^4y - 9x^2y^3 + 6xy^4$ and $6x^3y^2 + 24x^2y^3 - 30y^5$.

		Quotients
$ \begin{array}{r} 6y^2 \overline{) 6x^3y^2 + 24x^2y^3 - 30y^5} \\ \underline{x^3 + 4x^2y - 5y^3} \\ x^3 - 3xy^2 + 2y^3 \\ \underline{ 4x^2y + 3xy^2 - 7y^3} \\ 4x^2 + 3xy - 7y^2 \\ \underline{4x^2 - 4xy} \\ 7xy - 7y^2 \\ \underline{7xy - 7y^2} \\ 0 \end{array} $	$ \begin{array}{r} 3xy \overline{) 3x^4y - 9x^2y^3 + 6xy^4} \\ \underline{x^3 - 3xy^2 + 2y^3} \\ 4 \\ \underline{4x^3 - 12xy^2 + 8y^3} \\ 4x^3 + 3x^2y - 7xy^2 \\ \underline{- 3x^2y - 5xy^2 + 8y^3} \\ - 4 \\ \underline{12x^2y + 20xy^2 - 32y^3} \\ 12x^2y + 9xy^2 - 21y^3 \\ \underline{11y^2 \overline{) 11xy^2 - 11y^3}} \\ x - y \\ \underline{4x - 7y} \\ 0 \end{array} $	$ \begin{array}{r} 1 \\ x + 3y \\ 4x - 7y \end{array} $

H. C. F. = $3y(x - y)$.

Explanation. The monomial factors of the given expressions are $6y^2$ and $3xy$; their H. C. F., $3y$, is reserved as a factor of the answer. Remove from the second divisor ($4x^2y + 3xy^2 - 7y^3$) the simple factor y , and multiply the second dividend ($x^3 - 3xy^2 + 2y^3$) by 4. Similarly $-3x^2y - 5xy^2 + 8y^3$ is multiplied by -4 , and the factor $11y^2$ is removed from the last divisor $11xy^2 - 11y^3$.

EXERCISE

Find the H. C. F. of the following expressions:

1. $x^3 - 2x^2 + 4x + 7$ and $x^3 - 4x^2 + 10x - 7$.
2. $x^3 + 3x^2 - 3x - 5$ and $x^3 - 3x^2 + 2x + 6$.
3. $3x^3 + x^2 - x + 1$ and $15x^3 - x^2 - x + 3$.
4. $x^4 + x^2 + 1$ and $x^4 + x$.
5. $x^4 + x^2 + 1$ and $x^3y - y$.
6. $2x^3 - 5x^2 - 6x + 9$ and $3x^3 - 2x^2 + x - 2$.
7. $6x^3 + 13x^2 + 15x - 25$ and $2x^3 + 4x^2 + 4x - 10$.
8. $x^3 - 4x^2 + 2x + 1$ and $x^3 - 2x^2 + 3x - 2$.
9. $x^4 - 5x^3 + 5x^2 + 5x - 6$ and $x^3 - 3x^2 - 6x + 8$.
10. $a^4 + 4a^3 - 9a^2 - 16a + 20$ and $a^3 - 2a^2 - 23a + 60$.

11. $x^4 - 16x^3 + 86x^2 - 176x + 105$ and $x^4 - 19x^3 + 128x^2 - 356x + 336$.
12. $x^4 - 5x^3 + 5x^2 + 5x - 6$ and $x^4 + 2x^3 - 13x^2 - 14x + 24$.
13. $x^4 - 5ax^3 + 5a^2x^2 + 5a^3x - 6a^4$ and $x^3 + 4ax^2 + a^2x - 6a^3$.
14. $5x^3 - 9x^2 - 5x + 9$ and $2x^3 - x^2 - 2x + 1$.
15. $2x^3 - 2x^2 + 6x - 6$ and $2x^4 - 6x^3 + 10x^2 - 6x$.
16. $7x^3 - 4x^2y - 2xy^2 - y^3$ and $3x^3 - 3x^2y + xy^2 - y^3$.
17. $a^3 + 3a^2b - 5ab^2 + b^3$ and $4a^5 + 2a^4b - 8a^3b^2 + 2a^2b^3$.
18. $5a^4 + 23a^3b + 23ab^3 + 5b^4$ and $3a^4 + 14a^3b + 9ab^3 + 2b^4$.
19. $4x^4 + 7x^3 + 5x^2 - x - 3$ and $4x^4 + 5x^3 + 3x^2 - 2$.
20. $2x^4 - 8x^3 + x^2 + 11x + 3$ and $2x^4 - 4x^3 + x^2 + x - 3$.
21. $x^4 - x^3 - 3x^2 - 19x - 10$ and $x^4 - x^3 - 2x^2 - 17x - 5$.
22. $x^4 + 5x^3 + 11x^2 + 13x + 6$ and $x^3 - x^2 - 3x - 9$.

Reduce to lowest terms:

23. $\frac{2x^3 - x^2 - 16x + 15}{x^4 + 3x^3 - x - 3}$.

24. $\frac{x^3 + 3x^2 - 4x - 12}{x^4 + 5x^3 + 5x^2 - 5x - 6}$.

3. The Lowest Common Multiple of two expressions which cannot be factored by inspection can be found in the following manner:

Let A and B denote any two expressions whose H. C. F. is c , and let $\frac{A}{c} = a$ and $\frac{B}{c} = b$.

Then $A = a \cdot c, B = b \cdot c$.

Hence L. C. M. = $a \cdot c \cdot b$

$$= A \cdot \frac{B}{c}$$

To find the H. C. F. of the two expressions, divide one of the expressions by their H. C. F. and multiply the quotient by the other expression.

Thus, to find the L. C. M. of $x^3 - 2x^2 + x + 4$ and $x^3 - 3x^2 + 2x + 6$, determine their H. C. F. $(x + 1)$.

Dividing $x^3 - 2x^2 + x + 4$ by $x + 1$, we obtain $x^2 - 3x + 4$.

Hence, the L. C. M. $= (x^2 - 3x + 4)(x^3 - 3x^2 + 2x + 6)$.

EXAMPLES

Find the L. C. M. of the following expressions:

1. $x^3 + 7x^2 + 11x + 2$ and $x^3 + 8x^2 + 16x + 3$.
2. $a^3 + a^2 - a + 15$ and $a^3 + 2a^2 - 3a + 20$.
3. $3x^3 + x^2 + 3x + 5$ and $3x^3 + 7x^2 - x + 15$.
4. $2a^3 + 2a^2b - ab^2 + 6b^3$ and $2a^3 - 6a^2b + 7ab^2 - 6b^3$.
5. $2x^3 - 10x + 24$ and $x^3 + x^2 - 3x + 9$.
6. $a^4 + a^2 + 1$ and $a^4 - a^2 + 1$.
7. $a^3 + a - 2$ and $a^4 + a^3 - 2a - 4$.
8. $a^3 + ab^2 - 2b^3$ and $a^4 + a^2b - 2ab^3 - 4b^4$.
9. $a^6 - 6a^4 + 11a^2 - 6$ and $a^6 - 9a^4 + 26a^2 - 24$.
10. $a^2 - 2a - 3$, $a^3 + a^2 - 4a - 4$, $a^3 - 7a - 6$.
11. $a^3 - 2a^2 - 5a + 6$, $a^3 - 13a + 12$, $2a^3 - 11a^2 + 18a - 9$.

IV. CUBE ROOTS OF POLYNOMIALS AND ARITHMETICAL NUMBERS

1. By considering that the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$ is $a + b$, and employing a method similar to the one used for square roots, it can be shown that,

The first term a of the root is the cube root of the first term a^3 .

The second term of the root can be obtained by dividing $3a^2b$ by $3a^2$, the so-called trial divisor.

The complete divisor is obtained by dividing the last three terms by b , *i.e.* the complete divisor is $3a^2 + 3ab + b^2$.

The work may be arranged as follows:

$$\begin{array}{r} a^3 + 3a^2b + 3ab^2 + b^3 \mid a + b \\ a^3 \\ \hline \text{Trial divisor} = 3a^2 \mid 3a^2b + 3ab^2 + b^3 \\ \text{Complete divisor} = 3a^2 + 3ab + b^2 \mid 3a^2b + 3ab^2 + b^3 \end{array}$$

Ex. 1. Extract the cube root of $8x^3 - 27y^3 - 36x^2y + 54xy^2$.

$$\begin{array}{r} \text{Arranging,} \quad 8x^3 - 36x^2y + 54xy^2 - 27y^3 \mid 2x - 3y \\ 8x^3 \\ \hline 3a^2 = 12x^2 \quad -36x^2y + 54xy^2 - 27y^3 \\ 3ab + b^2 = -18xy + 9y^2 \\ 3a^2 + 3ab + b^2 = 12x^2 - 18xy + 9y^2 \mid -36x^2y + 54xy^2 - 27y^3 \end{array}$$

EXERCISE

Extract the cube root of the following expressions:

- $x^3 + 6x^2 + 12x + 8$.
- $8a^3 - 84a^2b + 294ab^2 - 343b^3$.
- $a^6 - 6a^5b + 12a^4b^2 - 8a^3b^3$.
- $64x^3 - 96x^2y + 48xy^2 - 8y^3$.
- $216a^6 - 756a^4b + 882a^2b^2 - 343b^3$.
- $1 + 3a + 6a^2 + 7a^3 + 6a^4 + 3a^5 + a^6$.
- $8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1$.
- $30x^2 - 12x + 8 - 25x^3 + 30x^4 - 12x^5 + 8x^6$.
- $8x^{2p} + 36x^{2p}y^{q+1} + 54x^py^{2q+2} + 27y^{3q+3}$.
- $8a^6 - 60a^5 + 114a^4 + 55a^3 - 171a^2 - 135a - 27$.
- $\frac{x^6}{a^6} + \frac{6x^4}{a^3} + 12x^2 + 8a^3$.
- $a^6 - a^3 + a^2 + \frac{1}{3} - \frac{2}{3a} + \frac{1}{3a^2} - \frac{1}{27a^3} + \frac{1}{9a^4} - \frac{1}{9a^5} + \frac{1}{27a^6}$.

2. The cube root of an arithmetical number is found in a similar manner.

Ex. 1. Extract the cube root of 6644.672.

Commencing at the decimal, divide the number into groups of digits, each group to contain 3 digits.

$$\begin{array}{r}
 6'644'.672 \quad | \quad 18.8 \\
 \begin{array}{l}
 3a^2 = 300 \\
 3ab = 240 \\
 b^2 = 64 \\
 604 \\
 3 \cdot 180^2 = 97200 \\
 3 \cdot 180 \cdot 8 = 4320 \\
 8^2 = 64 \\
 101584
 \end{array}
 \begin{array}{r}
 1 \\
 5644 \\
 4832 \\
 812672 \\
 812672
 \end{array}
 \end{array}$$

EXERCISE

Extract the cube roots of the following numbers:

- | | | |
|-------------|-------------|----------------|
| 1. 12167. | 5. 830584. | 9. .059319. |
| 2. 9261. | 6. 531441. | 10. 704969. |
| 3. 39304. | 7. 32.768. | 11. 43243551. |
| 4. 175.616. | 8. .274625. | 12. 237176659. |

Find to two decimal places the cube roots of the following numbers:

- | | | | |
|--------|--------|--------|---------|
| 13. 2. | 14. 5. | 15. 8. | 16. 10. |
|--------|--------|--------|---------|

V. INDETERMINATE EQUATIONS OF THE FIRST DEGREE

1. It was shown in § 189 that a linear equation involving two unknown quantities, such as $3x + 2y = 7$, has an infinite number of solutions. Similarly any system of equations is **indeterminate**, if the number of unknown quantities is greater than the number of equations. By introducing the condition that the roots shall be positive integers, the number of solutions can frequently be limited.

2. If the equation $ax + by = c$ (1)

is satisfied by the values $x=r_1$, and $y=r_2$, then it is also satisfied by the values:

$$\begin{cases} x = r_1 + mb, \\ y = r_2 - ma, \end{cases} \quad \text{or} \quad \begin{cases} x = r_1 - mb, \\ y = r_2 + ma. \end{cases}$$

For, substituting in (1), we obtain

$$a(r_1 \pm mb) + b(r_2 \mp ma) = c.$$

I.e. $ar_1 + br_2 = c.$

3. If one set of integers is known that satisfies an indeterminate equation involving two unknown quantities, all solutions can be found by the preceding paragraph.

Thus, the equation $3x + 2y = 34$ is satisfied by the values $x = 10$, $y = 2$. Hence it is satisfied by any values of the form:

$$x = 10 - 2m,$$

$$y = 2 + 3m.$$

E.g. If $m = 1$, $x = 8$, and $y = 5$.

If $m = 2$, $x = 6$, and $y = 8$, etc.

Ex. 1. Solve in positive integers:

$$7x + 16y = 209, \tag{1}$$

Dividing both members by the smaller coefficient, *i.e.* 7,

$$x + 2y + \frac{2y}{7} = 29 + \frac{6}{7}. \quad (2)$$

Since x and y , and hence $x + 2y$, are to be integers, the fractional parts of the two members must either be equal, or differ by an integral number.

Assume the first of these cases, if it produces an integral y :

$$\frac{2y}{7} = \frac{6}{7}, \quad \text{i.e. } y = 3.$$

Substituting in (2), $x + 6 = 29$, *i.e.* $x = 23$.

Hence if m denotes any integer, the general solutions are (§ 3):

$$x = 23 - 16m, \quad (3)$$

$$y = 3 + 7m. \quad (4)$$

Restricting the answers to positive values,

we have from (3), $m \leq 1$,

from (4), $m \geq 0$.

Therefore $m = 0$, or 1 .

Hence $x = 23$, or 7 ,

and $y = 3$, or 10 .

Ex. 2. Solve in positive integers:

$$43x + 20y = 798. \quad (1)$$

Dividing by 20, $2x + y + \frac{3x}{20} = 39 + \frac{18}{20}. \quad (2)$

Hence $\frac{3x}{20} = \frac{18}{20}$, *i.e.* $x = 6$.

Substituting in (2), $12 + y = 39$, *i.e.* $y = 27$.

Hence the general solutions are (§ 3):

$$x = 6 + 20m,$$

$$y = 27 - 43m.$$

To obtain positive values for x and y , m must be zero.

Hence $x = 6$ and $y = 27$ are the only solutions.

Ex. 3. Solve in positive integers :

$$23x + 37y = 3000. \quad (1)$$

Dividing by 23, $x + y + \frac{14y}{23} = 130 + \frac{10}{23}. \quad (2)$

Equating the fractional parts of the two members produces a fractional y , hence let

$$\frac{14y}{23} = \frac{10}{23} + n, \text{ where } n \text{ is an integer.}$$

Or $14y = 10 + 23n.$

This is another indeterminate equation, but a simpler one than (1). In complex cases, this equation may be treated by the regular method, while in simpler ones we find by trial a value of n which produces an integral y . Evidently, if $n = 2$, $10 + 23n$ is divisible by 14. Or, if

$$n = 2,$$

$$y = 4.$$

Equation (2) may be written,

$$x + y + n = 130.$$

Substituting, $x + 4 + 2 = 130$, *i.e.* $x = 124$.

Hence the general solutions are :

$$x = 124 - 37m, \quad (3)$$

$$y = 4 + 23m. \quad (4)$$

To make x positive, $m \leq 3$; to make y positive, $m \geq 0$.

Hence we have the following possible values :

$$m = 0, 1, 2, 3.$$

$$x = 124, 87, 50, 13.$$

$$y = 4, 27, 50, 73.$$

Ex. 4. Solve in positive integers :

$$6x - 17y = 27.$$

Dividing by 6, $x - 3y + \frac{y}{6} = 4 + \frac{3}{6}.$

Hence $y = 3$,
 and $x - 9 = 4$, *i.e.* $x = 13$.

Hence the general solutions are :

$$x = 13 - (-17)m,$$

or $x = 13 + 17m,$

and $y = 3 + 6m.$

By assigning to m any positive integral value, we obtain an unlimited number of solutions :

$$m = 0, 1, 2, 3, \dots,$$

$$x = 13, 30, 47, 64, \dots,$$

$$y = 3, 9, 15, 21, \dots$$

EXERCISE

Solve in positive integers :

- | | |
|------------------------|-------------------------|
| 1. $3x + 7y = 26.$ | 10. $2x + 13y = 37.$ |
| 2. $2x + 9y = 27.$ | 11. $7x + 48y = 1000.$ |
| 3. $7x + 16y = 62.$ | 12. $3x + 8y = 100.$ |
| 4. $7x + 11y = 151.$ | 13. $18x + 7y = 600.$ |
| 5. $38x + 7y = 216.$ | 14. $5x + 3y = 70.$ |
| 6. $9x + 29y = 179.$ | 15. $6x + 17y = 500.$ |
| 7. $47x + 11y = 1117.$ | 16. $3x + 11y = 100.$ |
| 8. $4x + 7y = 99.$ | 17. $5x + 7y + 4 = 56.$ |
| 9. $16x + 7y = 601.$ | 18. $46x + 41y = 2188.$ |

Solve in least possible positive integers :

- | | |
|----------------------|-----------------------|
| 19. $11x - 2y = 7.$ | 23. $3x - 8y = 10.$ |
| 20. $13x - 4y = 19.$ | 24. $5x - 7y = 3.$ |
| 21. $17x - 5y = 24.$ | 25. $7x - 18y = 20.$ |
| 22. $12x - 5y = 1.$ | 26. $14x - 45y = 11.$ |

Solve in positive integers :

$$27. \quad \begin{cases} x + 2y + 3z = 50. \\ 4x - 5y - 6z = -66. \end{cases} \qquad 28. \quad \begin{cases} x + 2y + 3z = 22. \\ 4x - 3y + 2z = 14. \end{cases}$$

29. Divide 142 into two parts such that one part is a multiple of 9, the other a multiple of 14.

30. Divide 1591 into two parts such that one part is a multiple of 23, the other of 34.

31. Find two fractions whose denominators are respectively 7 and 3, and whose sum equals $\frac{1}{2}\frac{6}{1}$.

32. In what manner can \$ 15 be paid in five-dollar bills and two-dollar bills ?

33. A farmer sold a number of horses and cows for \$447, receiving \$112 for each horse and \$37 for each cow. How many did he sell of each ?

34. A grocer bought a number of pounds of tea and coffee for \$5.10, paying for tea 40¢ per pound, and for coffee 19¢ per pound. How many pounds did he buy of each ?

35. A farmer sold a number of cows, sheep, and pigs for \$140, receiving \$31 for each cow, \$11 for each sheep, and \$9 for each pig. How many did he buy of each, if the total number of animals was 10 ?

VI. VARIATION

1. **Direct variation.** When the ratio of two variables is constant, each variable is said to *vary directly* as the other. Thus, x varies directly as y (or briefly, x varies as y), if $\frac{x}{y} = m$, a constant.

E.g. The weight of a quantity of water varies as its volume. The distance traversed by a man walking at a uniform rate varies as the time during which he walks, etc.

2. The sign of variation is \propto . It is read "varies as."

Thus, if $\frac{x}{y} = 3$, then $x \propto y$.

3. If $x \propto y$, then $x = my$, where m is a constant. For by definition, $\frac{x}{y}$ = a constant. Let this constant be m , then $x = my$.

NOTE. In the present chapter the letters m, n denote constants. If x varies as y , then $\frac{x}{y}$ is always taken equal to m .

Ex. 1. If $x \propto y$, and $x = 3$, when $y = 4$, find the value of x when $y = 10$.

Since $\frac{x}{y} = m$,

when $x = 3$, and $y = 4$, then $m = \frac{3}{4}$.

Hence $x = \frac{3}{4}y$.

When $y = 10$, $x = \frac{3}{4} \cdot 10 = 7\frac{1}{2}$.

4. Inverse variation. One quantity is said to vary inversely as another if it varies as the *reciprocal of the other*. Thus, x varies inversely as y , if $x \propto \frac{1}{y}$.

5. If x varies inversely as y , then xy is a constant.

Let $x = m \cdot \frac{1}{y}$, where m is a constant.

Hence we have $xy = m$.

6. Joint variation. One number is said to vary jointly as a number of others if it varies as their product.

Thus, x varies jointly as y and z if $x \propto yz$, or if $x = myz$.

Ex. 2. The volume v of a circular cone varies jointly as its height h , and the square of the radius r of its base. When $r = 10$, and $h = 12$, then $v = 1256$. Find v , if $h = 5$ and $r = 5$.

Let x be the required volume.

Since $v = mhr^2$, (1)

we have $1256 = m \cdot 12 \cdot 100$, (2)

and $x = m \cdot 5 \cdot 25$. (3)

We may find the value of m from (2), and substitute it in (3). It is, however, simpler to divide the members of (3) by those of (2).

$$\frac{x}{1256} = \frac{5 \cdot 25}{12 \cdot 100}.$$

Therefore

$$x = 130\frac{5}{8}.$$

EXERCISE

1. If $x \propto y$, and $x = 6$, when $y = 2$, find x when $y = 7$.
2. If $x \propto y$, and $x = 1$, when $y = 3$, find x when $y = 2$.
3. If $x \propto \frac{1}{y}$, and $x = 2$, when $y = 6$, find x when $y = 8$.
4. If $x \propto \frac{1}{y}$, and $x = 7$, when $y = 4$, find x when $y = 10$.
5. If $x \propto \frac{1}{y}$, and $x = 9$, when $y = 2$, find x , if $x = y$.
6. If x varies jointly as y and z ; and $x = 120$, when $y = 2$, $z = 15$; find x when $y = 4$, $z = 17$.
7. If $x \propto \frac{1}{y}$, and $x = 16$, when $y = 3$, find x when $y = 6$.
8. If $x \propto \frac{1}{y}$, and $x = 4a^2$, when $y = b^2$, find x when $y = 4ab$.
9. If $x \propto y^2$, and $x = 32$, when $y = 8$, find x when $y = 5$.
10. If $x \propto \frac{y}{z}$, and $x = 2$, when $y = 12$, $z = 2$; find x when $y = 7$, $z = 3$.
11. If $x \propto y$, and $y \propto z$, prove that $x \propto z$.
12. If $x \propto \frac{1}{y}$, and $y \propto \frac{1}{z}$, prove that $x \propto z$.
13. If $x \propto y$, prove that $x^2 \propto y^2$.
14. If $2a + 3b \propto 3a + b$, and $a = 2$, if $b = 1$, find a in terms of b .

15. If x varies directly as y^2 and inversely as z , and $x = 2$, when $y = 1, z = 3$; find the value of x , when $y = 2, z = 4$.

16. If $x \propto \frac{y}{z}$, and $x = 2$, when $y = 12, z = 2$; find x when $y = 7, z = 3$.

17. The area of a circle varies as the square of its radius. If the area of a circle is 100 square yards, when the radius is a feet, find the area of a circle whose radius is $3a$ feet.

18. The volume of a sphere varies as the cube of its diameter. If the volume of a sphere is 10 cubic inches, when its diameter is a inches, find the volume of a sphere whose diameter is $\frac{a}{2}$ inches.

19. The volume of a gas varies as the absolute temperature, and inversely as the pressure. The volume of a body of gas is 100 cubic inches when the pressure is 20 and the absolute temperature 250. What will be the volume when the pressure is 30 and the absolute temperature 360?

20. The illumination of an object varies inversely as the square of the distance from the source of light. If the illumination of an object at a distance of 5 feet from a source of light is 10, what is the illumination at a distance of 15 feet?

VII. LOGARITHMS

1. If $b^x = n$, (1)

then x is the **logarithm of n to the base b** , or in symbols

$$x = \log_b n. \quad (2)$$

Thus, since $2^3 = 8$, $3 = \log_2 8$;

since $3^4 = 81$, $4 = \log_3 81$.

2. The two equations (1) and (2) express the same relation, and *whenever we are unable to solve a problem stated in the form*

(2), we transform it to the form (1), i.e. we state it as an exponential equation.

Ex. 1. Determine x if $x = \log_5 125$.

Writing this equation in the exponential form, we have

$$5^x = 125.$$

Hence

$$x = 3.$$

Ex. 2. Find $\log_a 1$.

Let $x = \log_a 1$, then $a^x = 1$.

Hence $x = 0$, or $\log_a 1 = 0$.

3. Logarithms to the base 10 are known as **common logarithms**. If no base is written, 10 is understood as the base.

Thus, $\log 20 = \log_{10} 20$, $\log a = \log_{10} a$.

Ex. 3. Find $\log 100$.

Let $x = \log 100$; then $10^x = 100$.

Hence $x = 2$, or $\log 100 = 2$.

NOTE. The definition of logarithm can be expressed by the equation

$$b^{\log_b n} = n.$$

EXERCISE

Express the following relations as logarithmic equations:

- | | | |
|----------------------|-------------------------|----------------------------------|
| ✓ 1. $10^2 = 100$. | 4. $(\sqrt{2})^2 = 2$. | ✓ 7. $10^{-2} = \frac{1}{100}$. |
| ✓ 2. $10^3 = 1000$. | 5. $7^2 = 49$. | 8. $(\sqrt{3})^4 = 9$. |
| ✓ 3. $10^0 = 1$. | 6. $2^5 = 32$. | |

Express the following relations as exponential equations:

- | | | |
|---------------------------------|--------------------------|---------------------------------|
| 9. $3 = \log_4 64$. | ✓ 11. $-3 = \log .001$. | 13. $-3 = \log_2 \frac{1}{8}$. |
| ✓ 10. $6 = \log_{10} 1000000$. | ✓ 12. $x = \log a$. | ✓ 14. $\log 1 = 0$. |

Determine the following logarithms:

- | | | |
|---------------------|-----------------------------|------------------------|
| ✓ 15. $\log 1000.$ | 20. $\log_2 \frac{1}{2}.$ | 25. $\log_{16} 4.$ |
| ✓ 16. $\log 10000.$ | ✓ 21. $\log 100000.$ | 26. $\log_{27} 3.$ |
| 17. $\log_a a.$ | 22. $\log_3 243.$ | ✓ 27. $\log .00001.$ |
| 18. $\log_2 64.$ | ✓ 23. $\log \frac{1}{160}.$ | ✓ 28. $\log 10000000.$ |
| 19. $\log .1.$ | ✓ 24. $\log \sqrt[5]{10}.$ | |

Simplify:

- | | |
|-------------------------------|----------------------------------|
| 29. $2 \log_2 10 + \log_5 1.$ | 30. $\log_5 5^2 + \log_{11} 11.$ |
|-------------------------------|----------------------------------|

Solve the following equations:

- | | |
|---------------------|-------------------------|
| 31. $\log_3 x = 5.$ | 33. $\log_x 16 = 2.$ |
| 32. $\log_8 x = 2.$ | 34. $\log_x 10000 = 4.$ |

4. *The logarithm of 1 to any base is 0.*

Let $x = \log_b 1$, then $b^x = 1$, hence $x = 0$.

5. *The logarithm of the base is 1.*

Let $x = \log_b b$, then $b^x = b$, hence $x = 1$.

6. *The logarithm of a product is equal to the sum of the logarithms of its factors.*

I.e. if $\log_b m = x$, and $\log_b n = y$, (1)

then $\log_b mn = x + y$. (2)

Writing these equations in the exponential form (§ 2), we have:

If $m = b^x$, and $n = b^y$, then $mn = b^{x+y}$.

But in this form the conclusion is obvious. Substituting the values of x and y in equation (2), and omitting the bases, we have

$$\log (mn) = \log m + \log n.$$

E.g. $\log 15 = \log (3 \times 5) = \log 3 + \log 5.$

$$\log (abc) = \log a + \log (bc) = \log a + \log b + \log c.$$

7. *The logarithm of a quotient is equal to the logarithm of the dividend diminished by the logarithm of the divisor.*

I.e. if $\log_b m = x$, and $\log_b n = y$, (1)

then $\log_b \frac{m}{n} = x - y.$ (2)

This really means (§ 2):

If $m = b^x$ and $n = b^y$, then $\frac{m}{n} = b^{x-y}.$ (3)

But in this form the conclusion is obvious. Substituting the values of x and y in (2), and omitting the bases, we obtain

$$\log \frac{m}{n} = \log m - \log n.$$

E.g. $\log \frac{5}{2} = \log 5 - \log 2.$

$$\log \frac{15}{2} = \log \frac{3 \cdot 5}{2} = \log 3 + \log 5 - \log 2.$$

$$\log 5 = \log \frac{10}{2} = \log 10 - \log 2 = 1 - \log 2.$$

8. *The logarithm of a power of a number is equal to the logarithm of that number multiplied by the exponent of the power.*

I.e. if $\log_b n = x$, then $\log_b n^p = px.$ (1)

This means (§ 2):

If $n = b^x$, then $n^p = b^{px}.$ (2)

Again the conclusion is obvious.

Substituting the value of x in equation (1), and omitting the bases, we have

$$\log n^p = p \log n.$$

E.g. $\log 2^{10} = 10 \log 2$; $\log \frac{1}{8} = \log 2^{-3} = -3 \log 2$;

$$\log \frac{2 \times 7^3}{2^5} = \log 2 + 3 \log 7 - 5 \log 2$$

$$\log \frac{2^3 \times 3^2}{5^5 \times 6^2} = 3 \log 2 + 2 \log 3 - 5 \log 5 - 2 \log 6.$$

9. *The logarithm of a root of a number is equal to the logarithm of the number divided by the index.*

$$\log \sqrt[p]{n} = \log n^{\frac{1}{p}} = \frac{1}{p} \log n. \quad (\S 8)$$

E.g. $\log \sqrt[5]{7} = \frac{1}{5} \log 7.$

$$\log \frac{2^3 \cdot \sqrt[3]{3}}{5^2 \cdot \sqrt[7]{2}} = 3 \log 2 + \frac{1}{3} \log 3 - 2 \log 5 - \frac{1}{7} \log 2.$$

Ex. 1. Given $\log 2 = .30103$, and $\log 3 = .47712$, find $\log 288$.

$$\log 288 = \log 3^2 \cdot 2^5 = 2 \log 3 + 5 \log 2.$$

$$2 \log 3 = .95424$$

$$5 \log 2 = \underline{1.50515}$$

Therefore

$$\log 288 = 2.45939$$

Ex. 2. Given $\log 2 = .30103$, find $\log 25$.

$$\log 25 = 2 \log 5 = 2 \log \left(\frac{10}{2}\right) = 2 (\log 10 - \log 2).$$

But

$$\log 10 = 1.00000 \quad (\S 5)$$

$$\log 2 = \underline{.30103}$$

$$\log 10 - \log 2 = \underline{.69897}$$

Therefore

$$\log 25 = 1.39794.$$

EXERCISE

Given

$\log 2 = .30103$, $\log 3 = .47712$, $\log 5 = .69897$, $\log 7 = .84510$,
find the values of the following logarithms:

- | | | |
|-------------------------|---------------------------|--|
| 1. $\log 6$. | 8. $\log 2000$. | 15. $\log 4900$. |
| 2. $\log 14$. | 9. $\log 729$. | 16. $\log 225$. |
| 3. $\log 21$. | 10. $\log \frac{42}{5}$. | 17. $\log \frac{441}{25}$. |
| 4. $\log 12$. | 11. $\log \sqrt[3]{2}$. | 18. $\log \frac{2^3 \cdot 3^2 \cdot 5^2}{7^4}$. |
| 5. $\log \frac{2}{3}$. | 12. $\log 36$. | 19. $\log 2\frac{1}{4}$. |
| 6. $\log 8$. | 13. $\log 720$. | 20. $\log \sqrt[3]{16}$. |
| 7. $\log 30$. | 14. $\log 343$. | |

21. $\log \frac{\sqrt[4]{3}}{\sqrt[3]{2}}.$

24. $\log 2^{1000}.$

22. $\log \frac{\sqrt{2} \cdot \sqrt[3]{3} \cdot \sqrt[4]{5}}{\sqrt[6]{7}}.$

25. $\log .0027.$

23. $\log \sqrt[3]{\frac{480}{49}}.$

26. $\log \frac{.002}{.003^4}.$

Express in terms of $\log a$, $\log b$, $\log c$, and $\log d$:

27. $\log (a^2 b^3 c^5).$

31. $\log \frac{\sqrt[4]{a^3} \sqrt[5]{b^3}}{c^9}.$

28. $\log \frac{a^2 b^3 c^4}{d^2}.$

32. $\log \frac{a^{\frac{4}{3}} \sqrt[4]{b^5} \sqrt[3]{c} \sqrt[6]{d^5}}{\sqrt[6]{c^2 d}}.$

29. $\log \frac{a^4 \sqrt{b} \sqrt[3]{d}}{c^4}.$

33. $\log \frac{10 a^4 b^2 c}{\sqrt[5]{d}}.$

30. $\log \frac{\sqrt[4]{a} \sqrt[5]{b}}{\sqrt[6]{c}}.$

34. $\log \frac{100 \sqrt[4]{c^2}}{c^d}.$

Simplify:

35. $\log \frac{a^2 b^2}{c^2} + \log \frac{b^2 c^2}{d^2} + \log \frac{c^2 d^2}{a^2} - 2 \log bc.$

36. $\log (x+y) + \log (x-y) - \log (x^2 - y^2).$

37. $\log a + \log b + \log \frac{b}{c} + \log \frac{c}{d} - \log ab.$

38. $\log \frac{7000}{170} + \log \frac{17}{90}.$

39. $\log \left\{ \left(\frac{2}{35} \right)^{11} \div \frac{9^3}{7^{12}} \right\}.$

Solve the following equations:

40. $3 \log x = 2 \log 8.$

41. $\log x^2 - \log x = \log \frac{16}{x}.$

42. $\log 35 - \log x = \log 7.$

43. $\log 16x - \log 8x^2 = \log 8x^2 - 2 \log 4x.$

10. Properties of common logarithms. Since only a few numbers are perfect powers of 10, it is evident that common logarithms in general are not integral numbers.

The **characteristic** is the integral part of a logarithm. The **mantissa** is the fractional part of a logarithm.

E.g. in $\log 200 = 2.30103$, the characteristic is 2, and the mantissa is .30103.

11. The values of logarithms are usually so written that the mantissa is positive.

E.g. the value of $\log .03 = -1.52288$. But this equals $-2 + .47712$.

This is usually written: ·

$$\log .03 = .47712 - 2; \text{ or, } \log .03 = 8.47712 - 10.$$

NOTE. Some writers place the negative sign over the characteristic, to indicate that the characteristic alone is to be taken negative, thus $\log .03$ may be written $\bar{2}.47712$.

12. The characteristic of the logarithm of any number can be written by inspection.

$$\begin{aligned} (a) \quad 10^0 &= 1, & \text{hence } \log 1 &= 0. \\ 10^1 &= 10, & \text{hence } \log 10 &= 1. \\ 10^2 &= 100, & \text{hence } \log 100 &= 2. \\ 10^3 &= 1000, & \text{hence } \log 1000 &= 3, \text{ etc.} \end{aligned}$$

Therefore the logarithm of any number

between 1 and 10 = 0 + a fraction;

between 10 and 100 = 1 + a fraction;

between 100 and 1000 = 2 + a fraction; etc.

Hence it follows that:

The characteristic of the logarithm of any number greater than unity is positive, and is less by one than the number of digits in the integral part.

Thus, the characteristic of $\log 47.127$ is 1, of $\log 47216.2$ is 4, etc.

$$(b) \text{ Since } 10^0 = 1, \quad \text{we have } \log 1 = 0.$$

$$\text{Since } 10^{-1} = \frac{1}{10}, \quad \text{we have } \log .1 = -1.$$

$$\text{Since } 10^{-2} = \frac{1}{100}, \quad \text{we have } \log .01 = -2.$$

$$\text{Since } 10^{-3} = \frac{1}{1000}, \quad \text{we have } \log .0001 = -3, \text{ etc.}$$

.

Hence the characteristic of the logarithm of a number less than unity is negative. If no cipher immediately follows the decimal point, the number lies between 1 and .1, hence its logarithm $= -1 + \text{a fraction}$. If one cipher immediately follows the decimal point, the number lies between .1 and .01, hence its logarithm $= -2 + \text{a fraction}$, etc.

Hence we have in general:

If a number less than unity be expressed as a decimal, the characteristic of its logarithm is negative and numerically greater by one than the number of ciphers immediately following the decimal point.

E.g. the characteristic of $\log .476$ is -1 , of $\log .0123$ is -2 , of $\log .000047$ is -5 .

13. *If two numbers differ only in the position of the decimal point, their logarithms have equal mantissas, but different characteristics.*

Consider the numbers 4.1456 and .0041456.

$$4.1456 = 1000 \times .0041456.$$

$$\begin{aligned} \text{Hence } \log 4.1456 &= \log 1000 + \log .0041456 \\ &= 3 + \log .0041456. \end{aligned}$$

Hence the two logarithms differ by an integral number, *i.e.* their mantissas are equal.

14. The **table of logarithms** on pages 528 and 529 is only a table of mantissas, since the characteristics may be found by inspection.

15. **To find the logarithm of a number.** First write the characteristic by inspection. In determining the mantissa do not consider the decimal point. The mantissa is found from the table as follows :

(a) Let the given number consist of three figures, as 72.3. In the column headed *N* look for the first two significant figures (*i.e.* 72). The required mantissa is then found on the same horizontal line, in the column headed by the third figure (*i.e.* 3).

Thus, on a line with 72, in the column headed 3, we find 8591.

As the characteristic is 1, we have

$$\log 72.3 = 1.8591.$$

(b) If the number consists of less than three figures, add ciphers; thus, to find $\log 6$, find $\log 6.00$, $\log .57 = \log .570$, etc.

(c) Let the given number consist of more than three figures, as .064345.

The mantissa of $\log 64300 = .8082$.

The mantissa of $\log 64400 = .8089$.

That is, if the number increases 100, its logarithm increases .0007.

Hence, if the number 64300 increases 45, its logarithm will increase $\frac{45}{100}$ of .0007, or .0003.

Hence the mantissa $= .8082 + .0003 = .8085$.

Whence, $\log .064345 = .8085 - 2$ or $8.8085 - 10$.

16. The difference between two consecutive mantissas in the table is called the *tabular difference*.

N	O	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396
N	O	1	2	3	4	5	6	7	8	9

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996
N	0	1	2	3	4	5	6	7	8	9

In finding the logarithm of 64345 we assumed that the difference of two logarithms was proportional to the difference between the corresponding numbers, an assumption which gives approximately correct results if the difference between the numbers is small.

NOTE. For greater accuracy, tables of more places should be used. Five-place tables usually contain the mantissas of all numbers from 1 to 9999, and are constructed in the same manner as four-place tables, the only difference being that the first three figures are found in the column headed *N*, while the fourth one is at the top of the other columns.

If possible, the student should solve all examples of this chapter by means of a five-place table.

Ex. 1. Find $\log 2762400$.

The mantissa of $276 = .4409$.

Tabular difference = 16; $.24 \times 16 = 4$ (nearly).

Hence $\log 2762400 = 6.4413$.

Ex. 2. Find the $\log \frac{1414 \cdot 27^5}{.072 \cdot \sqrt[3]{102}}$.

Let $x = \frac{1414 \cdot 27^5}{.072 \cdot \sqrt[3]{102}}$.

Then $\log x = \log 1414 + 5 \log 27 - (\log .072 + \frac{1}{3} \log 102)$.

$\log 1414 = 3.1504$	$\log .072 = 8.8573 - 10$
$5 \log 27 = 7.1570$	$\frac{1}{3} \log 102 = .6695$
<u>10.3074</u>	<u>9.5268 - 10</u>
<u>9.5268 - 10</u>	
$\log x = 10.7806$	

EXERCISE

Verify the following statements:

- | | |
|---------------------------------|-----------------------------|
| 1. $\log 44 = 1.6435$. | 4. $\log 47210 = 4.6740$. |
| 2. $\log .002 = 7.30103 - 10$. | 5. $\log 14.566 = 1.1634$. |
| 3. $\log .791 = 9.8982 - 10$. | 6. $\log 1.8684 = 0.2714$. |

Find the logarithms of the following numbers:

- | | | | |
|------------|--------------|----------------|---------------|
| 7. 154. | 13. 4.2591. | 19. 70951. | 24. 506861. |
| 8. 2.34. | 14. .5271. | 20. 84.827. | 25. 21.1447. |
| 9. .0456. | 15. .06217. | 21. .00035995. | 26. .075907. |
| 10. 67100. | 16. .002457. | 22. 25.288. | 27. .0052208. |
| 11. 45.6. | 17. 45.72. | 23. 1.44598. | 28. 10134700 |
| 12. 1623. | 18. 18712. | | |

Find the logarithms of the following:

- | | |
|---|---|
| 29. $93 \times 3514.$ | 42. $\frac{47 \times .653 \times 12.83}{3576 \times 1520}.$ |
| 30. $1225 \times 387.$ | 43. $\frac{.765 \times .0018}{31457 \times 567.42}.$ |
| 31. $628 \times 493.$ | 44. $5^{27}.$ |
| 32. $3748 \times 1752 \times 4065.$ | 45. $16^{20}.$ |
| 33. $\frac{5}{4}.$ | 46. $(\frac{7}{8})^{14}.$ |
| 34. $\frac{3.9}{7}.$ | 47. $(\frac{1.5}{1.3})^{16}.$ |
| 35. $\frac{1.4}{3}.$ | 48. $(\frac{1.2}{5})^{32}.$ |
| 36. $15\frac{3}{4}.$ | 49. $\frac{.5936^{20} \times 386}{.076^{23}}.$ |
| 37. $7\frac{4}{13}.$ | 50. $\sqrt{5}.$ |
| 38. $\frac{1}{9}.$ | 51. $\sqrt{73567}.$ |
| 39. $\frac{319 \times 765}{138}.$ | 52. $\sqrt[3]{135}.$ |
| 40. $\frac{213 \times 7.655}{3145 \times 718}.$ | 53. $\sqrt[8]{15276}.$ |
| 41. $\frac{5.5347 \times 2.685}{137.65 \times 5944}.$ | 54. $\sqrt[5]{35107}.$ |
| | 55. $\sqrt[100]{13}.$ |

17. To find a number whose logarithm is given.

Ex. 1. To find x , if $\log x = 7.1931 - 10$.

Find 1931 in the table of mantissas. On the same line in the column headed N we find 15, the first two figures of x , and at the head of the column which contains 1931 is 6. Hence 1, 5, 6, are the figures of x . Since the characteristic is $7 - 10$, or -3 , x is a fraction, and two ciphers immediately follow the decimal point.

Whence $x = .00156$.

Ex. 2. Given $\log x = 1.0476$, required the value of x .

We find in the table the two mantissas, .0453 and .0492, corresponding with the numbers 111 and 112.

But $.0492 - .0453 = .0039$,

and $.0476 - .0453 = .0023$.

Hence if the mantissa .0453 increases .0039, then x increases 1.

Therefore, if the mantissa .0453 increases .0023, then x increases $\frac{23}{39}$.

Neglecting the decimal point, we have

$$x = 111\frac{23}{39} = 111.59.$$

But since the characteristic is 1, we have $x = 11.159$.

Ex. 3. Given $\log x = 8.4569 - 10$, required the value of x .

The next lower mantissa in the table is 4564, corresponding with the number 286. The difference of the two mantissas is 5, the tabular difference is 15, and $\frac{5}{15} = .33$.

Hence the figures of x are 28633.

Considering that the characteristic is -2 , we have

$$x = .028633.$$

EXERCISE

Find the numbers corresponding to the following logarithms:

- | | | |
|-----------------|---------------|-------------------|
| 1. 4.7796. | 5. .0170. | 9. .78134. |
| 2. .9430. | 6. .1038 - 3. | 10. 5.61658. |
| 3. .8949. | 7. 1.07426. | 11. 9.57938 - 10. |
| 4. 8.8376 - 10. | 8. 3.59478. | 12. 7.83145 - 10. |
13. 8.47652 - 10. 15. How many digits are in 2^{67} ?
14. 3.24567. 16. Find the number of digits in $2^{10} \cdot 3^{20} \cdot 4^{30}$.

18. Computation by Logarithms.—The approximate value of an arithmetical expression involving only multiplication, division, involution, and evolution may be found by logarithms.

If a greater logarithm has to be subtracted from a less, add to the minuend 10 - 10, *e.g.* for 2.4713 write 12.4713 - 10.

Ex. 1. Find the value of x , if

$$x = \frac{4.729 \times 3.214}{62.7 \times 8.392}.$$

$$\log x = \log 4.729 + \log 3.214 - (\log 62.7 + \log 8.392).$$

$$\log 4.729 = .6748$$

$$\log 62.7 = 1.7973$$

$$\log 3.214 = .5071$$

$$\log 8.392 = .9239$$

$$\begin{array}{r} 11.1819 - 10 \\ 2.7212 \\ \hline \end{array}$$

$$\begin{array}{r} 2.7212 \\ \hline \end{array}$$

$$\log x = 8.4607 - 10$$

Hence $x = .02889.$

Ex. 2. Find the value of x , if $x = .872^7$.

$$\log x = 7 \times \log .872$$

$$= 7 \times 9.9405 - 10$$

$$= 69.5835 - 70$$

$$= 0.5835 - 1.$$

Therefore

$$x = .38327.$$

19. To divide a logarithm whose characteristic is negative, write it in such a form that the negative portion of the characteristic is exactly divisible by the divisor. Thus, to divide $.4765 - 2$ by 3, write it in the form $1.4765 - 3$; to divide $8.4762 - 10$ by 7, write it in the form $12.4762 - 14$, or more conveniently $68.4762 - 70$.

Ex. 3. Find the value of x , if

$$x = \sqrt[12]{.1234}.$$

$$\begin{aligned}\log x &= \frac{1}{12} \log .1234 \\ &= \frac{1}{12} \cdot (9.0913 - 10) \\ &= \frac{1}{12} (119.0913 - 120) \\ &= 9.9243 - 10.\end{aligned}$$

Hence $x = .84$.

Ex. 4. Find the value of $\frac{-.0239 \times \sqrt[3]{41.3} \times .2^7}{42.3^2}$.

As negative numbers have no real logarithms, we determine first the value of the right member without regard to its sign.

$$\text{I.e. let } x = \frac{.0239 \times \sqrt[3]{41.3} \times .2^7}{42.3^2}.$$

$$\log x = \log .0239 + \frac{1}{3} \log 41.3 + 7 \log .2 - 2 \log 42.3.$$

$$\begin{array}{rcll}\log .0239 & = 8.3784 - 10 & = & 8.3784 - 10 \\ \frac{1}{3} \log 41.3 & = \frac{1}{3} (1.6160) & = & .5387 \\ 7 \log .2 & = 7 (9.3010 - 10) = & 5.1070 - 10 & \\ & & 14.0241 - 20 & \\ 2 \log 42.3 & = 2 (1.6263) & = & 3.2526 \\ & & \log x & = 10.7715 - 20\end{array}$$

$$x = .0000000005909.$$

Hence the required value $= -.0000000005909$.

20. The **cologarithm** of a number is equal to the negative value of the logarithm of that number. It is most conveniently obtained by subtracting the logarithm from 0, or from $10 - 10$.

$$E.g. \qquad \text{colog } 2 = 0 - \log 2$$

$$\text{But} \qquad \qquad \qquad 0 = 10 \qquad - 10$$

$$\begin{array}{r} \log 2 = \quad .30103 \\ \hline \text{colog } 2 = 9.69897 - 10 \end{array}$$

Hence, the cologarithm is obtained by subtracting every significant figure from 9 except the last, which is subtracted from 10, and annexing the negative characteristic -10 .

NOTE. Since $\log \frac{1}{x} = \log 1 - \log x = 0 - \log x$, the cologarithm of a number is also the logarithm of the reciprocal of the number.

21. The addition of a cologarithm is equivalent to the subtraction of a logarithm. Hence, examples may sometimes be simplified by the use of cologarithms. As a rule, however, the gain is very slight, and the beginner is advised not to use cologarithms until he is perfectly familiar with logarithms.

Ex. 5. Find the value of x , if

$$x = \frac{47.32 \times 41.92}{512.75}.$$

$$\log x = \log 47.32 + \log 41.92 + \text{colog } 512.75$$

$$\log 47.32 = 1.6751$$

$$\log 41.92 = 1.6224$$

$$\log 512.75 = 2.7099; \quad \text{colog } 512.75 = 7.2901 - 10$$

$$\log x \qquad = .5876$$

$$x \qquad = 3.869$$

EXERCISE

Find the value of the following expressions:*

1. 492.4×72.61 .
2. $6.417 \times .00342$.
3. $5.921 \times .072135$.
4. 8.927×794.25 .
5. $6249 \times .003217 \times .412$.
6. $37.21 \times 4729 \times .41115$.
7. $4.1929 \times (-4.789) \times 672$.
8. $.00423 \times (-.0472) \times (-42196)$.
9. $4719.1 \div 49.32$.
10. $.0048285 \div (-.08235)$.
11. $\frac{.6721 \times 4.238}{19.425}$.
12. $\frac{768.25 \times .49235}{.04216}$.
13. $\frac{421.75 \times .06255}{53.29 \times 1.9985}$.
14. $\frac{49876 \times .037542 \times 68.7075}{7.81649 \times 578.93 \times 28.4299}$.
15. $\frac{8.759236}{-.0576438}$.
16. $\frac{.000798543}{.000000965438}$.
17. $\frac{456.22 \times 72.555 \times 12^5}{103^3 \times 497 \times .0456789}$.
18. 1.357241^{10} .
19. 1.26677^{25} .
20. $.877058^9$.
21. 8095.371^{-3} .
22. $4\pi r^2$, if $\pi = 3.14159$ and $r = 2.0667$.
23. $214204^{\frac{7}{11}}$.
24. $39.679^{\frac{15}{4}}$.
25. $\left(\frac{3390 \times 4.3401}{13814.4}\right)^{11}$.
26. $.098756^{\frac{3}{7}}$.
27. $\sqrt[7]{8}$.
28. $\sqrt[4]{35246}$.
29. $\sqrt[12]{567348}$.
30. $\sqrt[6]{235.78}$.
31. $\sqrt[11]{3.1866}$.
32. $\sqrt[9]{1350\frac{7}{8}}$.
33. $(317.75)^6$.
34. $(\frac{1.67}{5.3})^{.32}$.
35. $2.71828^{4.60517}$.

* If possible, employ a five-place table.

$$36. \sqrt[17]{17^{1.2269}}.$$

$$37. \frac{99.1767^5 \times 12.34}{(20.358 \times 10.1575)^3}.$$

$$38. \frac{52072 \times \sqrt{.00734^9}}{.2556088}.$$

$$39. \sqrt[10]{1000} \div \sqrt[9]{100}.$$

$$40. \frac{7^7}{\sqrt[7]{7} \sqrt[7]{7}}.$$

$$41. \sqrt[10]{2 \sqrt[10]{2}}.$$

22. An **exponential equation** is an equation in which the unknown quantity occurs as an exponent, as $2^x = 7$. Such equations are readily solved by the use of logarithms.

Ex. 1. Find the value of x , if

$$23^x = 923.$$

Taking the logarithms of both members,

$$x \log 23 = \log 923.$$

$$\text{Hence} \quad x = \frac{\log 923}{\log 23} = \frac{2.9652}{1.3617} = 2.178.$$

23. **Change of system.** Logarithms to any base may be readily found by the preceding paragraph.

Ex. 2. Find $\log_7 12$.

$$\text{Let} \quad x = \log_7 12.$$

According to the principle of § 2, we have

$$7^x = 12.$$

Taking the logarithms of both members, and dividing,

$$x = \frac{\log 12}{\log 7}.$$

Ex. 3. Express $\log_b n$ by means of common logarithms.

$$\text{Let} \quad x = \log_b n;$$

$$\text{then} \quad b^x = n.$$

Taking the logarithm of both members, and dividing,

$$x = \frac{\log n}{\log b}.$$

EXERCISE

Solve the following equations:

- | | | |
|-------------------|------------------------------------|-----------------------------|
| 1. $4^x = 18$. | 5. $(.5)^x = .10$. | 9. $\sqrt[x]{10} = 2$. |
| 2. $5^x = 16$. | 6. $10^{x+1} = 2^x$. | 10. $\sqrt[x]{14.5} = 3$. |
| 3. $14^x = 224$. | 7. $4^x \cdot 5^x = 1700$. | 11. $\sqrt[3x]{1000} = 5$. |
| 4. $7^x = 370$. | 8. $4^{x+1} \cdot 5^{x-1} = 100$. | 12. $\sqrt[x]{20} = 10^x$. |

Find to three places of decimals:

- | | | |
|------------------|-------------------|-------------------|
| 13. $\log_7 8$. | 15. $\log_4 3$. | 17. $\log_4 19$. |
| 14. $\log_2 9$. | 16. $\log_3 77$. | 18. $\log_7 66$. |

19. If $\frac{ab^x - 1}{b - 1} = c$, find (a) the value of x in terms of a , b , and c ; (b) the numerical value of x , if $a = 5$, $b = 10$, and $c = 11$.

20. The n th term of a geometric series is l , the first term is a , and the common ratio is r . Find n in terms of a , l , and r .

21. Prove that $\log_b a \cdot \log_a b = 1$.

VIII. COMPOUND INTEREST AND ANNUITIES

1. To find the amount a_n of a principal of p dollars for n years at $R\%$ compound interest.

Let $R\%$ or $\frac{R}{100} = r$.

Then

$$a_1 = p(1 + r),$$

$$a_2 = a_1(1 + r) = p(1 + r)^2,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$a_n = p(1 + r)^n. \tag{1}$$

Taking the logarithm of both members, we obtain

$$\log a_n = \log p + n \log (1 + r). \tag{2}$$

Equations (1) and (2) may be used to find any of the four quantities a_n , p , r , and n , if the other three are given.

Ex. 1. Find the amount of \$1200 for 5 years at 4% compound interest.

Substituting the given values in (2),

$$\begin{aligned}\log a_5 &= \log 1200 + 5 \log 1.04 \\ &= 3.07918 + 5 \times .01703 = 3.16433.\end{aligned}$$

Hence $a_5 = \$1459.93$.

Ex. 2. In how many years will \$100 amount to \$1100 at 5% compound interest?

Substituting in (2),

$$\log 1100 = \log 100 + n \log 1.05.$$

$$\begin{aligned}\text{Hence } n &= \frac{\log 1100 - 2}{\log 1.05} \\ &= \frac{1.04139}{.02119} = 49.1 \text{ years.}\end{aligned}$$

NOTE. Since formula (1) is only true for integral values of n , the fractional part of the preceding answer is only an approximation.

2. The **compound interest** I is the difference between amount and principal, or:

$$I = p(1 + r)^n - p. \quad (3)$$

3. If the interest is compounded semiannually, the amount a_n' is obtained by the method of § 1.

$$a_n' = p \left(1 + \frac{r}{2} \right)^{2n}. \quad (4)$$

4. An **annuity** is a fixed sum of money, payable at equal intervals of time.

5. To find the amount A_n of an annuity of s dollars left unpaid for n years at $R\%$ compound interest.

The s dollars due at the end of the first year will amount in the remaining $n - 1$ years to $s(1 + r)^{n-1}$ dollars. Similarly the payment due at the end of the second year will at the end of the entire period amount to $s(1 + r)^{n-2}$ dollars, etc.

$$\text{Hence} \quad A_n = s(1 + r)^{n-1} + s(1 + r)^{n-2} + \cdots + s.$$

Adding the G. P. (§ 365)

$$A_n = \frac{s}{r} [(1 + r)^n - 1]. \quad (5)$$

Again, this formula may be used to find any of the four quantities A , s , r , and n , if the other three are known.

6. To find the present value A_0 of an annuity of s dollars to continue for n years at $R\%$ compound interest.

At the end of n years A_0 would amount to $A_0(1 + r)^n$.

The annuity at the same time would amount to $\frac{s}{r} [(1 + r)^n - 1]$.

$$\text{Hence} \quad A_0(1 + r)^n = \frac{s}{r} [(1 + r)^n - 1].$$

$$\text{Therefore} \quad A_0 = \frac{s}{r} \left[1 - \frac{1}{(1 + r)^n} \right]. \quad (6)$$

Ex. 3. Find the present value of an annuity of \$ 800, for 20 years, allowing compound interest at 4% per annum.

$$A_0 = \frac{800}{.04} \left[1 - \frac{1}{1.04^{20}} \right],$$

$$\begin{aligned} \log \left(\frac{1}{1.04^{20}} \right) &= 0 - 20 \log 1.04 \\ &= 0 - .3406 = .6594 - 1. \end{aligned}$$

$$\frac{1}{(1.04)^{20}} = .45644.$$

$$\text{Hence} \quad A_0 = \frac{434.848}{.04} = \$ 10871.20.$$

EXERCISE

Find the amount at compound interest

1. Of \$ 200 for 8 years, at $4\frac{1}{2}\%$.
2. Of \$ 1550 for 5 years, at 5% .
3. Of \$ 1 for 1000 years, at 4% .
4. Of \$ 20 for 20 years, at 2% , interest compounded semi-annually.
5. Find the compound interest of \$ 250 for 10 years at 6% compound interest.
6. Find the principal that will amount to \$ 1000 in 20 years at $4\frac{1}{2}\%$ compound interest.
7. Find the principal that will amount to \$ 420 in 10 years at 5% compound interest.
8. In what time will \$ 8007 amount to \$ 21218 at $4\frac{3}{4}\%$ compound interest?
9. In what time will \$ 1 amount to \$ 5000000 at $5\frac{9}{10}\%$ compound interest?
10. At what rate will \$ 200 in 10 years amount to \$ 350 at compound interest?
11. At what rate will \$ 40 in 12 years amount to \$ 80, interest compounded semiannually?
12. Find the value of an annuity of \$ 40 left unpaid for 10 years, 5% compound interest being allowed.
13. Find the present value of an annuity of \$ 800 for 10 years, 4% compound interest being allowed.
14. Find the present value of an annuity of \$ 500 for 4 years, 5% compound interest being allowed.
15. Find the present value of an annuity of \$ 900 for 12 years, 5% compound interest being allowed.
16. What annuity can be purchased for \$ 2500, if it is to run for 10 years, and 6% annual compound interest is allowed?

IX. SUMMATION OF SERIES

1. The sum of $f(1) + f(2) + f(3) \cdots + f(n)$ is frequently denoted by $\sum_1^n x f(x)$ or $\sum_1^n f(x)$.*

Thus,
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \sum_1^n (x^2).$$

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_1^n \frac{1}{x}.$$

$$\sqrt{1+1} + \sqrt{1+3} + \sqrt{1+3} + \cdots + \sqrt{1+n} = \sum_1^n \sqrt{1+x}.$$

$$\log 5 + \log 6 + \log 7 + \log 8 = \sum_5^8 \log x.$$

2. If the lower limit is omitted, 1 is understood, and if the upper limit is omitted, n is understood.

Thus,
$$\sum (x^3) = \sum_1^4 (x^3) = 1^3 + 2^3 + 3^3 + 4^3.$$

$$\Sigma f(x) = f(1) + f(2) + f(3) + \cdots + f(n).$$

$$\Sigma(1+a)^x = (1+a) + (1+a)^2 + (1+a)^3 + \cdots + (1+a)^n.$$

3.
$$\Sigma[f(x) + F(x)] = \Sigma f(x) + \Sigma F(x). \quad (1)$$

For,
$$\begin{aligned} \Sigma[f(x) + F(x)] &= (f(1) + F(1)) + (f(2) + F(2)) \\ &\quad + (f(3) + F(3)) + \cdots + (f(n) + F(n)) \\ &= [f(1) + f(2) + f(3) + \cdots + f(n)] \\ &\quad + [F(1) + F(2) + F(3) + \cdots + F(n)] \\ &= \Sigma f(x) + \Sigma F(x). \end{aligned}$$

Thus,
$$\Sigma(x^2 + x) = \Sigma(x^2) + \Sigma(x).$$

$$\Sigma\left(3x^2 + 2x - \frac{1}{x}\right) = \Sigma(3x^2) + \Sigma(2x) - \Sigma\left(\frac{1}{x}\right).$$

$$\sum_1^3 \left(x + \frac{1}{x}\right) = \sum_1^3 (x) + \sum_1^3 \left(\frac{1}{x}\right) = 1 + 2 + 3 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} = 7\frac{5}{6}.$$

* Σ is the Greek letter Sigma.

4. If m denotes a constant, then

$$\Sigma mf(x) = m \Sigma f(x). \quad (2)$$

For,

$$\begin{aligned} \Sigma mf(x) &= mf(1) + mf(2) + mf(3) + \dots + mf(n) \\ &= m[f(1) + f(2) + f(3) + \dots + f(n)] \\ &= m \Sigma f(x). \end{aligned}$$

Thus,

$$\Sigma(3x^4) = 3 \Sigma(x^4).$$

$$\Sigma(3x^4 + 2x^3) = \Sigma(3x^4) + \Sigma(2x^3) = 3 \Sigma(x^4) + 2 \Sigma(x^3).$$

$$\sum_1^4 (73x) = 73 \sum_1^4 (x) = 73(1 + 2 + 3 + 4) = 730.$$

5. If the quantity that follows the summation symbol (Σ) does not contain a variable, all terms of the sum are equal.

Thus,

$$\Sigma(1) = 1 + 1 + 1 + \dots + 1 = n.$$

$$\Sigma(5) = 5 + 5 + 5 + \dots \text{to } n \text{ terms} = 5n.$$

Ex. 1. Write the series which is represented by $\Sigma(x+1)(x+5)$.

$$\Sigma(x+1)(x+5) = 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 8 + \dots + (n+1)(n+5).$$

Ex. 2. Write the following series in the abbreviated form:

$$1 \cdot 3 \cdot 5 + 2 \cdot 4 \cdot 6 + 3 \cdot 5 \cdot 7 + \dots + n(n+2)(n+4).$$

Evidently the series $= \Sigma x(x+2)(x+4)$.

Ex. 3. Find the numerical value of $\sum_1^4 (x \cdot 2^x)$.

$$\begin{aligned} \sum_1^4 (x \cdot 2^x) &= 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 \\ &= 2 + 8 + 24 + 64 = 98. \end{aligned}$$

Ex. 4. Simplify $\Sigma(2x^2 + 3)^2$.

$$\begin{aligned} \Sigma(2x^2 + 3)^2 &= \Sigma(4x^4 + 12x^2 + 9) \\ &= \Sigma(4x^4) + \Sigma(12x^2) + \Sigma 9 \\ &= 4 \Sigma(x^4) + 12 \Sigma(x^2) + 9n. \end{aligned}$$

Ex. 5. If $\Sigma f(x) = 12$, and $\Sigma F(x) = 2\frac{1}{2}$, find $\Sigma[3f(x) - 5F(x)]$.

$$\begin{aligned}\Sigma[3f(x) - 5F(x)] &= \Sigma(3f(x)) - \Sigma(5F(x)) \\ &= 3\Sigma f(x) - 5\Sigma F(x) \\ &= 3 \cdot 12 - 5 \cdot \frac{5}{2} = 23\frac{1}{2}.\end{aligned}$$

EXERCISE

Write the series which are represented by the following symbols:

- | | |
|--|--|
| 1. $\Sigma \frac{1}{x}$. | 8. Σa^x . |
| 2. $\Sigma (x+1)(x+2)(x+3)$. | 9. $\Sigma x \cdot a^x$. |
| 3. $\Sigma (\log x)$. | 10. $\Sigma (2x+1)m^x$. |
| 4. $\Sigma (x^x)$. | 11. $\Sigma (y^2 a^y)$. |
| 5. $\Sigma \frac{x(x+1)}{x^2}$. | 12. $\Sigma {}^k C_x$. |
| 6. $\Sigma (a+x)$. | 13. $\sum_{\downarrow} {}^4 C_x$. |
| 7. $\Sigma \left(x^2 + \frac{1}{x}\right)$. | 14. $\sum_0^6 {}^{10} C_x \cdot a^x$. |

Write the following series in the abbreviated form:

15. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1)$.
16. $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \dots + n(n+1)^2$.
17. $\frac{1 \cdot 2}{3 \cdot 4} + \frac{2 \cdot 3}{4 \cdot 5} + \frac{3 \cdot 4}{5 \cdot 6} + \dots + \frac{n(n+1)}{(n+2)(n+3)}$.
18. $2 + 4 + 6 + 8 + \dots + 2n$.
19. $1 + 3 + 5 + 7 + \dots + (2n-1)$.
20. $1 \cdot 2 \cdot 3 + 2 \cdot 4 \cdot 5 + 3 \cdot 6 \cdot 7 + \dots + n(2n)(2n+1)$.
21. $1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + 4 \cdot 9 + \dots$ to n terms.
22. $1 \cdot 2 \cdot 1 + 2 \cdot 4 \cdot 3 + 3 \cdot 6 \cdot 5 + 4 \cdot 8 \cdot 7 + \dots$ to n terms.

23. $\frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$ to n terms.

24. $1 + 2a + 3a^2 + 4a^3 + \dots$ to n terms.

25. $\frac{100}{1} + \frac{100 \cdot 99}{1 \cdot 2} + \frac{100 \cdot 99 \cdot 98}{\underline{1 \cdot 2 \cdot 3}} + \dots$ to n terms.

Find the numerical value of:

26. $\sum_1^3 \frac{1}{x^2}$.

29. $\sum_1^3 (x+1)(x+2)$.

27. $\sum_1^5 x^2$.

30. $\sum_2^6 (x^2 + x)$.

28. $\sum_1^{11} (7)$.

31. $\sum_1^5 \frac{1}{x+2}$.

Simplify:

32. $\sum (4x + 2)$.

36. $\sum x(x+1)^2$.

33. $\sum (9x - 7)$.

37. $\sum_1^6 [f(x) - f(x-1)]$.

34. $\sum (2x^2 + 3x + 5)$.

38. $\sum [f(x) - f(x-1)]$.

35. $\sum \left(3x^3 + 5x - \frac{6}{x} \right)$.

If $\sum (x) = \frac{n(n+1)}{2}$, and $\sum (x^2) = \frac{n(n+1)(2n+1)}{6}$, find:

39. $\sum (2x + 1)$.

43. $\sum (x+1)(x+2)$.

40. $\sum (3x - 2)$.

44. $\sum (x+4)^2$.

41. $\sum (x^2 - 2x)$.

45. $1 \cdot 3 + 2 \cdot 5 + 3 \cdot 7 + \dots$
 $+ n(2n+1)$.

42. $\sum (2x^2 - 4x + 9)$.

Prove that:

46. $\sum_1^n f(x) - \sum_1^{n-1} f(x) = f(n)$.

47. $\sum [f(x+1) - f(x)] = f(n+1) - f(1)$.

48. $\sum_{2n} [f(x+1) - f(x-1)] = f(n+1) + f(n) - f(1) - f(0)$.

$$6. \Sigma[f(x) - f(x-1)] = f(n) - f(0). \quad (3)$$

$$\begin{aligned} \text{For,} \quad \Sigma[f(x) - f(x-1)] &= f(1) - f(0) \\ &\quad + f(2) - f(1) \\ &\quad + f(3) - f(2) \\ &\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ &\quad + f(n-1) - f(n-2) \\ &\quad + f(n) - f(n-1). \end{aligned}$$

But this obviously equals $f(n) - f(0)$.

7. Formula (3) can be used to determine the sum of a number of series.

Ex. 1. Let $f(x) = x^2$.

Substituting in (3), we obtain

$$\Sigma[x^2 - (x-1)^2] = n^2.$$

Simplifying,

$$\Sigma(2x - 1) = n^2.$$

$$2 \Sigma(x) - n = n^2.$$

Hence

$$\Sigma(x) = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}.$$

I.e.

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}.$$

Ex. 2. Let $f(x) = x^3$.

Substituting in (3), we obtain

$$\Sigma[x^3 - (x-1)^3] = n^3.$$

Simplifying,

$$\Sigma(3x^2 - 3x + 1) = n^3.$$

Or,

$$3 \Sigma(x^2) - 3 \Sigma(x) + n = n^3.$$

Substituting the value of $\Sigma(x)$, (Ex. 1), and transposing,

$$3 \Sigma(x^2) = n^3 + \frac{3n(n+1)}{2} - n.$$

$$\Sigma(x^2) = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(n+1)(2n+1)}{6}.$$

$$I.e. \quad 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

$$E.g. \quad 1^2 + 2^2 + 3^2 + \dots + 80^2 = \frac{80 \cdot 81 \cdot 161}{6} = 173880.$$

8. In a similar manner the sum of the first n natural numbers raised to any power may be found, provided the sums of the lower powers are known.

To find $\Sigma(x^3)$ make $f(x) = x^4$, to find $\Sigma(x^4)$ make $f(x) = x^5$, etc.

9. The sum $\Sigma f(x)$, if $f(x)$ is a rational integral function, can be reduced to the sum of the following series:

$$\Sigma(x) = \frac{n(n+1)}{2}, \quad (4)$$

$$\Sigma(x^2) = \frac{n(n+1)(2n+1)}{6}, \quad (5)$$

$$\Sigma(x^3) = \frac{n^2(n+1)^2}{4}, \text{ etc.} \quad (6)$$

Ex. 3. Find the value of $\Sigma(12x^2 - 4x + 3)$.

$$\begin{aligned} \Sigma(12x^2 - 4x + 3) &= 12 \cdot \frac{n(n+1)(2n+1)}{6} - 4 \cdot \frac{n(n+1)}{2} + 3n \\ &= 4n^3 + 4n^2 + 3n. \end{aligned}$$

Ex. 4. Find the sum of the series

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \dots \text{ to } n \text{ terms.}$$

$$\begin{aligned} \text{The series} &= \Sigma x(x+1)(x+2) \\ &= \Sigma(x^3 + 3x^2 + 2x) \\ &= \Sigma(x^3) + 3\Sigma(x^2) + 2\Sigma(x) \\ &= \frac{n^2(n+1)^2}{4} + \frac{3 \cdot n(n+1)(2n+1)}{6} + n(n+1) \\ &= \frac{n(n+1)}{4} [n^2 + 5n + 6] \\ &= \frac{1}{4} n(n+1)(n+2)(n+3). \end{aligned}$$

EXERCISE

1. Prove that $\Sigma(x^3) = \left[\frac{n(n+1)}{2} \right]^2$.
2. Prove that $\Sigma(x^4) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{2 \cdot 3 \cdot 5}$.

Find the sum of the following series:

3. $1^3 + 2^3 + 3^3 \dots + 100^3$.
4. $\sum_{10} (x^4)$.
5. $\Sigma(3x-1)$.
6. $\Sigma(6x^2 + 4x + 1)$.
7. $\Sigma(3x^2 + 2x - 1)$.
8. $\Sigma(x+1)(3x+2)$.
9. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots$ to n terms.
10. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \dots$ to n terms.
11. $3 \cdot 7 + 4 \cdot 9 + 5 \cdot 11 + \dots$ to n terms.
12. $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \dots + n(n+1)^2$.
13. $1 \cdot 2 \cdot 2 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 6 + \dots + n(n+1)2n$.
14. $1 \cdot 2 \cdot 3 \cdot 4 + 2 \cdot 3 \cdot 4 \cdot 5 + 3 \cdot 4 \cdot 5 \cdot 6 \dots$ to n terms.
15. Find the sum of the first n odd numbers.
16. Find the sum of the first n even numbers.
17. Find $11^2 + 12^2 + 13^2 + \dots$ to 40 terms.
18. Find $\sum_{50}^{100} (x^3)$.

10. Arithmetic series. Let

$$f(x) = {}^x C_k = \frac{x(x-1) \dots (x-k+1)}{[k]}$$

Then $f(x-1) = {}^{x-1} C_k$

and $f(x) - f(x-1) = {}^{x-1} C_{k-1}$ (§ 381).

Substituting in the general formula (3),

$$\Sigma^{x-1} C_{k-1} = {}^n C_k.$$

By substituting for k respectively the values 2, 3, 4, we obtain

$$\Sigma^{x-1} C_1 = \Sigma(x-1) = \frac{n(n-1)}{2}, \quad (7)$$

$$\Sigma^{x-1} C_2 = \Sigma \frac{(x-1)(x-2)}{2} = \frac{n(n-1)(n-2)}{3}, \quad (8)$$

$$\Sigma^{x-1} C_3 = \Sigma \frac{(x-1)(x-2)(x-3)}{3} = \frac{n(n-1)(n-2)(n-3)}{4}. \quad (9)$$

$$\text{I.e. } 0 + 1 + 2 + \dots + (n-1) = \frac{n(n-1)}{2},$$

$$0 + 0 + 1 + 3 + 6 + \dots = \frac{n(n-1)(n-2)}{3}, \text{ etc.}$$

11. If, in a series, each term is subtracted from the following one, we obtain the series of **first differences**. If each term of the series of first differences is subtracted from the following one, we obtain the series of the **second differences**, etc.

Thus, consider the series	1	3	9	22	45	81
First differences,	2	6	13	23	36	
Second differences,		4	7	10	13	
Third differences,			3	3	3	
Fourth differences,				0	0	

12. An **arithmetic series of the n th order** is a series whose n th differences are all equal.

Thus, an ordinary arithmetic progression may be considered an arithmetic series of the first order. The series considered in § 11 is an arithmetic series of the third order, etc.

13. If we consider the following series and its first differences,

Series, $a_1 \quad a_2 \quad a_3 \quad \dots \quad a_x$

First differences, $b_1 \quad b_2 \quad b_{x-1}$

it is obvious that

$$a_x = a_1 + (b_1 + b_2 + \dots + b_{x-1}). \quad (10)$$

14. In an arithmetic series of the first order the first differences are all equal, hence

$$a_x = a_1 + (x-1)b_1.$$

If we denote the sum of the first n terms of a series by s_n , we obtain

$$s_n = \Sigma(a_x).$$

Hence, for an arithmetic progression, we have

$$s_n = \Sigma[a_1 + (x-1)b],$$

$$s_n = na_1 + \frac{n(n-1)}{2}b_1. \quad (11)$$

NOTE. These results were obtained in slightly different form in Chapter XXI.

15. Let an arithmetic series of the second order and its two series of differences be represented as follows:

Series, $a_1 \quad a_2 \quad a_3 \quad \dots \quad a_x$

First differences, $b_1 \quad b_2 \quad \dots \quad b_{x-1}$

Second differences, $c_1 \quad \dots \quad c_1$

Then $a_x = a_1 + (b_1 + b_2 \dots b_{x-1}).$ (§ 13)

But $b_1 + b_2 \dots b_{x-1}$ is an arithmetic series of the first order, which may be added by formula (11). Hence, making $n = x-1$, we have

$$a_x = a_1 + (x-1)b_1 + \frac{(x-1)(x-2)}{2}c_1.$$

But $s_n = \Sigma a_x$

$$= \Sigma \left[a_1 + b_1 \Sigma (x-1) + c_1 \Sigma \frac{(x-1)(x-2)}{2} \right].$$

Or, considering § 10,

$$s_n = na_1 + \frac{n(n-1)}{2} b_1 + \frac{n(n-1)(n-2)}{3} c_1.$$

16. Similarly, we may treat the arithmetic series of the third order.

Series,	a_1	a_2	a_3	a_4	a_5	\dots	a_x
First differences,	b_1	b_2	b_3	b_4	\dots	b_{x-1}	
Second differences,		c_1	c_2	c_3	\dots	c_{x-2}	
Third differences,			d_1	d_1	\dots	d_1	

$$a_x = a_1 + (b_1 + b_2 + \dots + b_{x-1}) \quad (\S 13)$$

$$= a_1 + (x-1)b_1 + \frac{(x-1)(x-2)}{2} c_1$$

$$+ \frac{(x-1)(x-2)(x-3)}{3} d_1. \quad (\S 15)$$

And $s_n = \Sigma a_x$

$$= na_1 + b_1 \Sigma (x-1) + c_1 \Sigma \frac{(x-1)(x-2)}{2}$$

$$+ d_1 \Sigma \frac{(x-1)(x-2)(x-3)}{3}.$$

Or, $s_n = na_1 + \frac{n(n-1)}{2} b_1 + \frac{n(n-1)(n-2)}{3} c_1$

$$+ \frac{n(n-1)(n-2)(n-3)}{4} d_1.$$

17. By mathematical induction it can be shown that these results are true for arithmetic series of any order. Omitting the subscript 1 of the first term of each series of differences, we have in general:

$$a_x = a_1 + (x-1)b + \frac{(x-1)(x-2)}{2}c + \dots$$

$$s_n = na_1 + \frac{n(n-1)}{2}b + \frac{n(n-1)(n-2)}{3}c + \dots$$

Ex. Find the 10th term and the sum of the first 8 terms of the series 1, 3, 8, 20, 43, 81.

Series,	1	3	8	20	43	81
First differences,	2	5	12	23	38	
Second differences,		3	7	11	15	
Third differences,			4	4	4	

Hence, $a = 1, b = 2, c = 3, d = 4.$

$$a_{10} = 1 + 9 \cdot 2 + \frac{9 \cdot 8}{1 \cdot 2} \cdot 3 + \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} \cdot 4 = 463.$$

$$s_8 = 8 \cdot 1 + \frac{8 \cdot 7}{1 \cdot 2} \cdot 2 + \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3} \cdot 3 + \frac{8 \cdot 7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 4} \cdot 4 = 512.$$

EXERCISE

- Find the 11th term of 3, 6, 11, 18, 27, ...
- Find the 9th term of 2, 6, 12, 20, 30, ...
- Find the 10th term of 2, 9, 28, 65, 126, 217, ...
- Find the 7th term of 2, 3, 7, 15, 28, 47, ...
- Find the 20th term of 1, 4, 10, 23, 47, ...
- Find the sum of the series 1, 4, 10, 23, 47, ... to 10 terms.
- Find the sum of the series 10, 28, 56, 94, 142, 200, ... to n terms.

8. Find the sum of the series 3, 23, 71, 159, 299, 503, ... to 8 terms.

9. Find the sum of the series 1, 1+2, 1+2+3, 1+2+3+4, ... to 10 terms.

10. Find the sum of the series 1^2 , $1^2 + 2^2$, $1^2 + 2^2 + 3^2$, $1^2 + 2^2 + 3^2 + 4^2$, ... to 9 terms.

11. Show that the series represented by $\Sigma(2x^2 - 4x + 9)$ is an arithmetic series of the second order.

12. Show that the series represented by $\Sigma(x^3 - 5x)$ is an arithmetic series of the third order.

18. Many **fractional series** can be added by means of formula (3). Values of $f(x)$ which contain the factor x in the denominator, however, should be avoided, since $f(x-1)$ would become ∞ , if $x=1$.

Ex. 1. Let $f(x) = \frac{1}{x+1}$.

Substituting in (3), $\Sigma\left(\frac{1}{x+1} - \frac{1}{x}\right) = \frac{1}{n+1} - 1$.

Simplifying, $\Sigma \frac{1}{x(x+1)} = \frac{n}{n+1}$.

I.e. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$.

19. Similarly, $f(x) = \frac{1}{(x+1)(x+2)}$

leads to the value of $\Sigma \frac{1}{x(x+1)(x+2)}$,

$$f(x) = \frac{1}{(x+1)(x+2)(x+3)}$$

leads to $\Sigma \frac{1}{x(x+1)(x+2)(x+3)}$, etc.

Ex. 2. Find the sum of the series

$$\frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{1}{3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \dots \text{ to } n \text{ terms.}$$

According to § 19, let $f(x) = \frac{1}{(x+1)(x+2)(x+3)(x+4)}$.

Substituting in (3), and simplifying, we obtain

$$\Sigma \frac{1}{x(x+1)(x+2)(x+3)(x+4)} = \frac{1}{96} - \frac{1}{4(n+1)(n+2)(n+3)(n+4)}.$$

20. To add a series of the form

$$\frac{1}{a(a+b)} + \frac{1}{(a+b)(a+2b)} + \frac{1}{(a+2b)(a+3b)} + \dots \text{ to } n \text{ terms,}$$

let
$$f(x) = \frac{1}{a+bx}.$$

Thus, to find $\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} \dots$ to n terms, let $f(x) = \frac{1}{2+3x}$.

Substituting in (3),

$$\Sigma \left(\frac{1}{2+3x} - \frac{1}{3x-1} \right) = \frac{1}{2+3n} - \frac{1}{2}.$$

Hence,
$$\Sigma \frac{1}{(3x-1)(3x+2)} = \frac{n}{2(2+3n)}.$$

I.e.
$$\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} + \dots \text{ to } n \text{ terms} = \frac{n}{2(2+3n)}.$$

21. The series $\frac{1}{1 \cdot 4} + \frac{1}{2 \cdot 5} + \frac{1}{3 \cdot 6} + \frac{1}{4 \cdot 7} \dots$ is equal to three

series of the preceding kind, viz.

$$\begin{aligned} & \left(\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10} \dots \right) + \left(\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} \dots \right) \\ & \quad + \left(\frac{1}{3 \cdot 6} + \frac{1}{6 \cdot 9} + \frac{1}{9 \cdot 12} \dots \right). \end{aligned}$$

22. An infinite series is **convergent** if the sum of its first n terms approaches a finite limit, as n is increased indefinitely.

The limit of the sum thus obtained is called the **sum** of the infinite series.

Thus, $a_1 + a_2 + a_3 + \cdots + a_n$ or $\Sigma(a_x)$ is convergent if $\lim \left[\sum_1^n (a_x) \right]_{n=\infty}$ is a finite number. The value of $\sum_1^n (a_x)_{n=\infty}$ or $\sum_1^\infty (a_x)$ is the sum of the infinite series.

E.g. the series $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} \cdots$ to infinity, is convergent, and its sum equals 1, for $\lim \left(\frac{n}{n+1} \right)_{n=\infty} = 1$. (§ 18)

Similarly, $\sum_1^\infty \frac{1}{x(x+1)(x+2)(x+3)(x+4)} = \frac{1}{96}$, and the series is convergent (§ 19).

EXERCISE

Find the sum to n terms, and to an infinite number of terms, of the following series:

$$1. \quad \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \cdots$$

$$2. \quad \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{1}{3 \cdot 4 \cdot 5 \cdot 6} + \cdots$$

$$3. \quad \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots$$

$$4. \quad \frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \frac{1}{6 \cdot 8} + \cdots$$

$$5. \quad \frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \cdots, \text{ if } n \text{ is even.}$$

$$6. \quad \frac{1}{3 \cdot 7} + \frac{1}{7 \cdot 11} + \frac{1}{11 \cdot 15} + \cdots$$

$$7. \frac{1}{1 \cdot 7} + \frac{1}{4 \cdot 10} + \frac{1}{7 \cdot 13} + \dots, \text{ if } n \text{ is even.}$$

$$8. \frac{1}{a(a+b)} + \frac{1}{(a+b)(a+2b)} + \frac{1}{(a+2b)(a+3b)} + \dots.$$

$$9. \frac{1}{2 \cdot 3 \cdot 4} + \frac{2}{3 \cdot 4 \cdot 5} + \frac{3}{4 \cdot 5 \cdot 6} + \dots.$$

HINT. Let $f(x) = \frac{x(x+1)}{(x+2)(x+3)}.$

10. By substituting $f(x) = ar^x$, derive the formula for the sum of a geometric progression, viz. Σar^{x-1} . (§ 366)

11. By substituting $f(x) = x \cdot r^x$, derive the formula :

$$\Sigma xr^{x-1} \text{ or } 1 + 2r + 3r^2 + 4r^3 + \dots = \frac{nr^{n+1} - (n+1)r^n + 1}{(r-1)^2}.$$

12. Find $1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \dots$ to n terms.

13. Find $1 + 2 \cdot 5 + 3 \cdot 5^2 + 4 \cdot 5^3 + \dots$ to 11 terms.

14. Find $1 \cdot r + 2r^2 + 3r^3 + 4r^4 + \dots$ to n terms.

15. Find $1 - 2 \cdot 2 + 3 \cdot 2^2 - 4 \cdot 2^3 + \dots$ to n terms.

16. Find $1 \cdot 1 + 2 \cdot 1 \cdot 2 + 3 \cdot 1 \cdot 2 \cdot 3 \dots + n|n.$

HINT. Let $f(x) = \underline{|x+1}.$

Derive a series by substituting in formula (3).

$$17. f(x) = \frac{x(3x+1)}{(x+1)(x+2)}.$$

$$20. f(x) = \frac{x(x+1)}{(2x+1)(2x+3)}.$$

$$18. f(x) = \frac{x(x+1)}{\underline{|2}}.$$

$$21. f(x) = \frac{1}{\underline{|x+1}}.$$

$$19. f(x) = \frac{x(x+1)(x+2)}{\underline{|3}}.$$

$$22. f(x) = \frac{x(x+1)}{(x+4)(x+5)}.$$

$$23. f(x) = x^2 r^x.$$

NOTE. Students who are familiar with trigonometry may apply formula (3) to trigonometric functions.

E.g. by substituting $f(x) = \sin(A + Bx)$, we obtain

$$\Sigma \cos [A + (x - \frac{1}{2})B] = \frac{\sin \frac{n}{2} B \cos \left(A + \frac{n}{2} B \right)}{\sin \frac{B}{2}}.$$

Or, substituting $C - \frac{B}{2}$ for A ,

$$\Sigma \cos [C + (x - 1)B] = \frac{\sin \frac{n}{2} B \cos \left(C + \frac{n-1}{2} B \right)}{\sin \frac{B}{2}}.$$

$$*E.g.* \quad \cos B + \cos 2 B + \cos 3 B + \dots + \cos n B = \frac{\sin \frac{n}{2} B \cos \frac{n+1}{2} B}{\sin \frac{B}{2}}.$$

Or,

$$\cos B + \cos 3 B + \cos 5 B + \dots + \cos(2n-1)B = \frac{\sin 2nB}{2 \sin B}.$$

$$\text{Or,} \quad \cos \frac{\pi}{n} + \cos \frac{2\pi}{n} + \cos \frac{3\pi}{n} + \dots + \cos \frac{n\pi}{n} = -1.$$

Similarly,

$$\sin B + \sin 2 B + \sin 3 B + \dots + \sin n B = \frac{\sin \frac{n}{2} B \sin \frac{n+1}{2} B}{\sin \frac{B}{2}}.$$

$$\sin B + \sin 3 B + \sin 5 B + \dots + \sin(2n-1)B = \frac{\sin^2 nB}{\sin B}.$$

$$\sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \sin \frac{3\pi}{n} + \dots + \sin \frac{n\pi}{n} = \cot \frac{\pi}{2n}.$$

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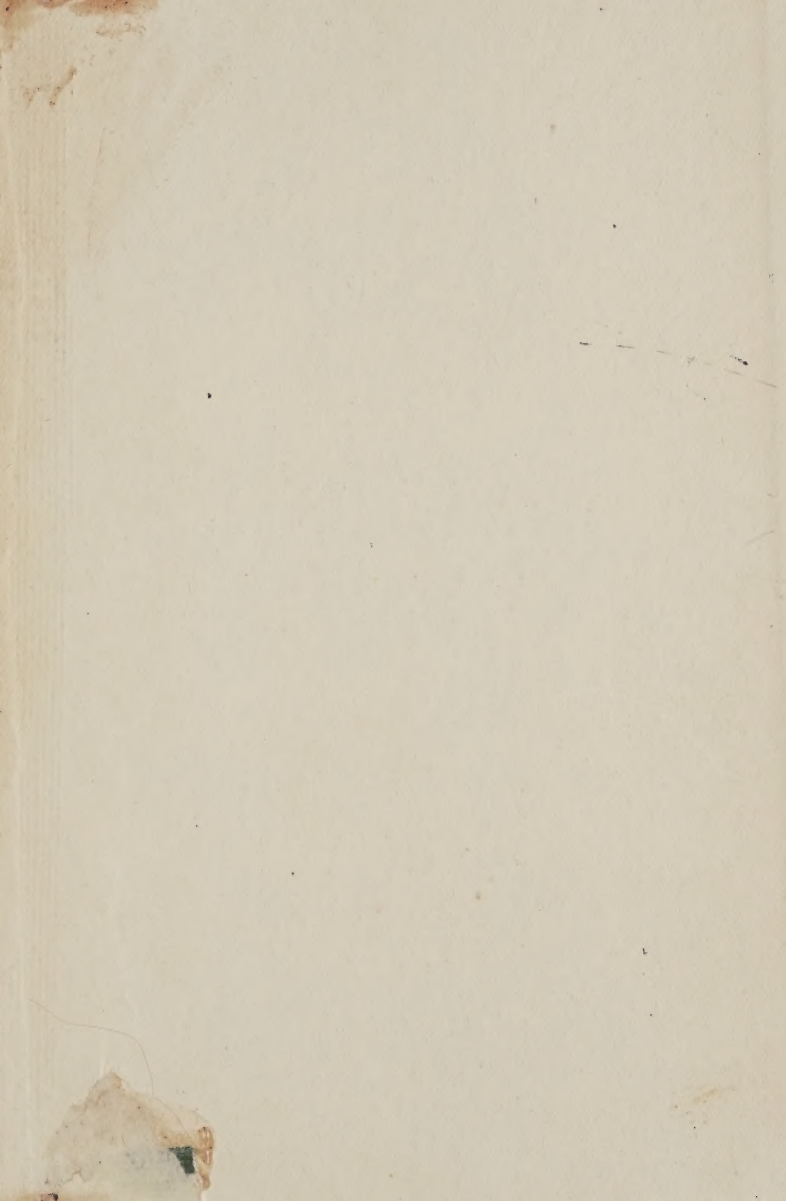
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